A First Course in
Complex Analysis

Version 1.24

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These are the lecture notes of a one-semester undergraduate course which we have taught several times at Binghamton University (SUNY) and San Francisco State University. For many of our students, complex analysis is their first rigorous analysis (if not mathematics) class they take, and these notes reflect this very much. We tried to rely on as few concepts from real analysis as possible. In particular, series and sequences are treated "from scratch." This also has the (maybe disadvantageous) consequence that power series are introduced very late in the course.

We thank our students who made many suggestions for and found errors in the text. Special thanks go to Collin Bleak, Jon Clauss, Sharma Pallekonda, and Joshua Palmatier for comments after teaching from this book.

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Chapter 1

Complex Numbers

Die ganzen Zahlen hat der liebe Gott geschaffen, alles andere ist Menschenwerk.
(God created the integers, everything else is made by humans.)
Leopold Kronecker (1823–1891)

1.1   Definition and Algebraic Properties

The *complex numbers* can be defined as pairs of real numbers,

$$\mathbb{C} = \{(x, y) : x, y \in \mathbb{R}\},$$

equipped with the *addition*

$$(x, y) + (a, b) = (x + a, y + b)$$

and the *multiplication*

$$(x, y) \cdot (a, b) = (xa - yb, xb + ya).$$

One reason to believe that the definitions of these binary operations are "good" is that $\mathbb{C}$ is an extension of $\mathbb{R}$, in the sense that the complex numbers of the form $(x, 0)$ behave just like real numbers; that is, $(x, 0) + (y, 0) = (x + y, 0)$ and $(x, 0) \cdot (y, 0) = (x \cdot y, 0)$. So we can think of the real numbers being embedded in $\mathbb{C}$ as those complex numbers whose second coordinate is zero.

The following basic theorem states the algebraic structure that we established with our definitions. Its proof is straightforward but nevertheless a good exercise.

Theorem 1.1. $(\mathbb{C}, +, \cdot)$ *is a* field*; that is:*

$$\forall\, (x, y), (a, b) \in \mathbb{C} : \; (x, y) + (a, b) \in \mathbb{C} \tag{1.1}$$

$$\forall\, (x, y), (a, b), (c, d) \in \mathbb{C} : \; \big((x, y) + (a, b)\big) + (c, d) = (x, y) + \big((a, b) + (c, d)\big) \tag{1.2}$$

$$\forall\, (x, y), (a, b) \in \mathbb{C} : \; (x, y) + (a, b) = (a, b) + (x, y) \tag{1.3}$$

$$\forall\, (x, y) \in \mathbb{C} : \; (x, y) + (0, 0) = (x, y) \tag{1.4}$$

$$\forall\, (x, y) \in \mathbb{C} : \; (x, y) + (-x, -y) = (0, 0) \tag{1.5}$$

$$\forall\, (x, y), (a, b), (c, d) \in \mathbb{C} : \; (x, y) \cdot \big((a, b) + (c, d)\big) = (x, y) \cdot (a, b) + (x, y) \cdot (c, d)) \tag{1.6}$$

$$\forall\,(x,y),(a,b) \in \mathbb{C}:\ (x,y)\cdot(a,b) \in \mathbb{C} \tag{1.7}$$

$$\forall\,(x,y),(a,b),(c,d) \in \mathbb{C}:\ \big((x,y)\cdot(a,b)\big)\cdot(c,d) = (x,y)\cdot\big((a,b)\cdot(c,d)\big) \tag{1.8}$$

$$\forall\,(x,y),(a,b) \in \mathbb{C}:\ (x,y)\cdot(a,b) = (a,b)\cdot(x,y) \tag{1.9}$$

$$\forall\,(x,y) \in \mathbb{C}:\ (x,y)\cdot(1,0) = (x,y) \tag{1.10}$$

$$\forall\,(x,y) \in \mathbb{C}\setminus\{(0,0)\}:\ (x,y)\cdot\left(\tfrac{x}{x^2+y^2}, \tfrac{-y}{x^2+y^2}\right) = (1,0) \tag{1.11}$$

Remark. What we are stating here can be compressed in the language of algebra: equations (1.1)–(1.5) say that $(\mathbb{C},+)$ is an *Abelian group* with *unit element* $(0,0)$, equations (1.7)–(1.11) that $(\mathbb{C}\setminus\{(0,0)\},\cdot)$ is an abelian group with unit element $(1,0)$. (If you don't know what these terms mean—don't worry, we will not have to deal with them.)

The definition of our multiplication implies the innocent looking statement

$$(0,1)\cdot(0,1) = (-1,0). \tag{1.12}$$

This identity together with the fact that

$$(a,0)\cdot(x,y) = (ax, ay)$$

allows an alternative notation for complex numbers. The latter implies that we can write

$$(x,y) = (x,0) + (0,y) = (x,0)\cdot(1,0) + (y,0)\cdot(0,1).$$

If we think—in the spirit of our remark on the embedding of $\mathbb{R}$ in $\mathbb{C}$—of $(x,0)$ and $(y,0)$ as the real numbers x and y, then this means that we can write any complex number (x,y) as a linear combination of $(1,0)$ and $(0,1)$, with the real coefficients x and y. $(1,0)$, in turn, can be thought of as the real number 1. So if we give $(0,1)$ a special name, say i, then the complex number that we used to call (x,y) can be written as $x\cdot 1 + y\cdot i$, or in short,

$$x + iy.$$

The number x is called the *real part* and y the *imaginary part*[1] of the complex number $x+iy$, often denoted as $\mathrm{Re}(x+iy) = x$ and $\mathrm{Im}(x+iy) = y$. The identity (1.12) then reads

$$i^2 = -1.$$

We invite the reader to check that the definitions of our binary operations and Theorem 1.1 are coherent with the usual real arithmetic rules if we think of complex numbers as given in the form $x+iy$.

1.2 Geometric Properties

Although we just introduced a new way of writing complex numbers, let's for a moment return to the (x,y)-notation. It suggests that one can think of a complex number as a two-dimensional real vector. When plotting these vectors in the plane $\mathbb{R}^2$, we will call the x-axis the *real axis* and the y-axis the *imaginary axis*. The addition that we defined for complex numbers resembles vector addition. The analogy stops at multiplication: there is no "usual" multiplication of two vectors
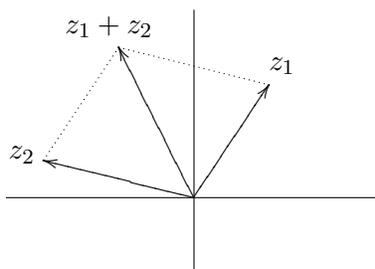
Figure 1.1: Addition of complex numbers.

that gives another vector—much less so if we additionally demand our definition of the product of two complex numbers.

Any vector in $\mathbb{R}^2$ is defined by its two coordinates. On the other hand, it is also determined by its length and the angle it encloses with, say, the positive real axis; let's define these concepts thoroughly. The *absolute value* (sometimes also called the *modulus*) of $x + iy$ is

$$r = |x + iy| = \sqrt{x^2 + y^2}\,,$$

and an *argument* of $x + iy$ is a number ϕ such that

$$x = r \cos \phi \qquad \text{and} \qquad y = r \sin \phi\,.$$

This means, naturally, that any complex number has many arguments; more precisely, all of them differ by a multiple of 2π.

The absolute value of the difference of two vectors has a nice geometric interpretation: it is the *distance* of the (end points of the) two vectors (see Figure 1.2). It is very useful to keep this geometric interpretation in mind when thinking about the absolute value of the difference of two complex numbers.
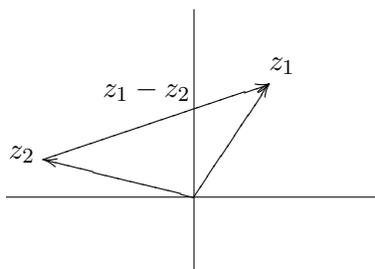
Figure 1.2: Geometry behind the "distance" between two complex numbers.

The first hint that absolute value and argument of a complex number are useful concepts is the fact that they allow us to give a geometric interpretation for the multiplication of two complex numbers. Let's say we have two complex numbers, $x_1 + iy_1$ with absolute value r_1 and argument ϕ_1, and $x_2 + iy_2$ with absolute value r_2 and argument ϕ_2. This means, we can write

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[1]The name has historical reasons: people thought of complex numbers as unreal, imagined.

$x_1 + iy_1 = (r_1 \cos \phi_1) + i(r_1 \sin \phi_1)$ and $x_2 + iy_2 = (r_2 \cos \phi_2) + i(r_2 \sin \phi_2)$ To compute the product, we make use of some classic trigonometric identities:

$$
\begin{aligned}
(x_1 + iy_1)(x_2 + iy_2) &= \big((r_1 \cos \phi_1) + i(r_1 \sin \phi_1)\big)\big((r_2 \cos \phi_2) + i(r_2 \sin \phi_2)\big) \\
&= (r_1 r_2 \cos \phi_1 \cos \phi_2 - r_1 r_2 \sin \phi_1 \sin \phi_2) + i(r_1 r_2 \cos \phi_1 \sin \phi_2 + r_1 r_2 \sin \phi_1 \cos \phi_2) \\
&= r_1 r_2 \big((\cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2) + i(\cos \phi_1 \sin \phi_2 + \sin \phi_1 \cos \phi_2)\big) \\
&= r_1 r_2 \big(\cos(\phi_1 + \phi_2) + i \sin(\phi_1 + \phi_2)\big).
\end{aligned}
$$

So the absolute value of the product is $r_1 r_2$ and (one of) its argument is $\phi_1 + \phi_2$. Geometrically, we are multiplying the lengths of the two vectors representing our two complex numbers, and adding their angles measured with respect to the positive x-axis.[2]
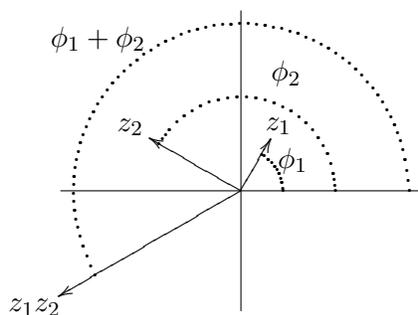
Figure 1.3: Multiplication of complex numbers.

In view of the above calculation, it should come as no surprise that we will have to deal with quantities of the form $\cos \phi + i \sin \phi$ (where ϕ is some real number) quite a bit. To save space, bytes, ink, etc., (and because "Mathematics is for lazy people"[3]) we introduce a shortcut notation and define
$$ e^{i\phi} = \cos \phi + i \sin \phi . $$

At this point, this exponential notation is indeed purely a notation. We will later see that it has an intimate connection to the complex exponential function. For now, we motivate this maybe strange-seeming definition by collecting some of its properties. The reader is encouraged to prove them.

Lemma 1.2. *For any* $\phi, \phi_1, \phi_2 \in \mathbb{R}$,

(a) $e^{i\phi_1} e^{i\phi_2} = e^{i(\phi_1 + \phi_2)}$

(b) $1/e^{i\phi} = e^{-i\phi}$

(c) $e^{i(\phi + 2\pi)} = e^{i\phi}$

(d) $\left| e^{i\phi} \right| = 1$

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[2]One should convince oneself that there is no problem with the fact that there are many possible arguments for complex numbers, as both cosine and sine are periodic functions with period 2π.

[3]Peter Hilton (Invited address, Hudson River Undergraduate Mathematics Conference 2000)

(e) $\frac{d}{d\phi} e^{i\phi} = i\, e^{i\phi}$.

With this notation, the sentence "The complex number $x+iy$ has absolute value r and argument ϕ" now becomes the identity

$$x + iy = re^{i\phi}.$$

The left-hand side is often called the *rectangular form*, the right-hand side the *polar form* of this complex number.

From very basic geometric properties of triangles, we get the inequalities

$$-|z| \leq \operatorname{Re} z \leq |z| \qquad \text{and} \qquad -|z| \leq \operatorname{Im} z \leq |z|\,. \tag{1.13}$$

The square of the absolute value has the nice property

$$|x + iy|^2 = x^2 + y^2 = (x + iy)(x - iy)\,.$$

This is one of many reasons to give the process of passing from $x + iy$ to $x - iy$ a special name: $x - iy$ is called the *(complex) conjugate* of $x + iy$. We denote the conjugate by

$$\overline{x + iy} = x - iy\,.$$

Geometrically, conjugating z means reflecting the vector corresponding to z with respect to the real axis. The following collects some basic properties of the conjugate. Their easy proofs are left for the exercises.

Lemma 1.3. *For any* $z, z_1, z_2 \in \mathbb{C}$,

(a) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

(b) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

(c) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

(d) $\overline{\overline{z}} = z$

(e) $|\overline{z}| = |z|$

(f) $|z|^2 = z\overline{z}$

(g) $\operatorname{Re} z = \frac{1}{2}\left(z + \overline{z}\right)$

(h) $\operatorname{Im} z = \frac{1}{2i}\left(z - \overline{z}\right)$

(i) $\overline{e^{i\phi}} = e^{-i\phi}$.

From part (f) we have a neat formula for the inverse of a non-zero complex number:

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}\,.$$

A famous geometric inequality (which holds for vectors in $\mathbb{R}^n$) is the *triangle inequality*

$$|z_1 + z_2| \leq |z_1| + |z_2|\,.$$

By drawing a picture in the complex plane, you should be able to come up with a geometric proof of this inequality. To prove it algebraically, we make extensive use of Lemma 1.3:

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2 \\
&= |z_1|^2 + 2\operatorname{Re}(z_1\overline{z_2}) + |z_2|^2.
\end{aligned}$$

Finally by (1.13)

$$\begin{aligned}
|z_1 + z_2|^2 &\leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \\
&= |z_1|^2 + 2|z_1||\overline{z_2}| + |z_2|^2 \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2,
\end{aligned}$$

which is equivalent to our claim.

For future reference we list several variants of the triangle inequality:

Lemma 1.4. *For $z_1, z_2, \cdots \in \mathbb{C}$, we have the following identities:*

(a) *The triangle inequality:* $|\pm z_1 \pm z_2| \leq |z_1| + |z_2|$.

(b) *The reverse triangle inequality:* $|\pm z_1 \pm z_2| \geq |z_1| - |z_2|$.

(c) *The triangle inequality for sums:* $\left| \sum_{k=1}^{n} z_k \right| \leq \sum_{k=1}^{n} |z_k|$.

The first inequality is just a rewrite of the original triangle inequality, using the fact that $|\pm z| = |z|$, and the last follows by induction. The reverse triangle inequality is proved in Exercise 15.

1.3   Elementary Topology of the Plane

In Section 1.2 we saw that the complex numbers $\mathbb{C}$, which were initially defined algebraically, can be identified with the points in the Euclidean plane $\mathbb{R}^2$. In this section we collect some definitions and results concerning the topology of the plane. While the definitions are essential and will be used frequently, we will need the following theorems only at a limited number of places in the remainder of the book; the reader who is willing to accept the topological arguments in later proofs on faith may skip the theorems in this section.

Recall that if $z, w \in \mathbb{C}$, then $|z - w|$ is the distance between z and w as points in the plane. So if we fix a complex number a and a positive real number r then the set of z satisfying $|z - a| = r$ is the set of points at distance r from a; that is, this is the circle with center a and radius r. The *inside* of this circle is called the *open disk* with center a and radius r, and is written $D_r(a)$. That is, $D_r(a) = \{z \in \mathbb{C} : |z - a| < r\}$. Notice that this does not include the circle itself.

We need some terminology for talking about subsets of $\mathbb{C}$.

Definition 1.1. Suppose E is any subset of $\mathbb{C}$.

(a) A point a is an *interior* point of E if some open disk with center a lies in E.

(b) A point b is a *boundary* point of E if every open disk centered at b contains a point in E and also a point that is not in E.

(c) A point c is an *accumulation* point of E if every open disk centered at c contains a point of E different from c.

(d) A point d is an *isolated* point of E if it lies in E and some open disk centered at d contains no point of E other than d.

The idea is that if you don't move too far from an interior point of E then you remain in E; but at a boundary point you can make an arbitrarily small move and get to a point inside E and you can also make an arbitrarily small move and get to a point outside E.

Definition 1.2. A set is *open* if all its points are interior points. A set is *closed* if it contains all its boundary points.

Example 1.1. For $R > 0$ and $z_0 \in \mathbb{C}$, $\{z \in \mathbb{C} : |z - z_0| < R\}$ and $\{z \in \mathbb{C} : |z - z_0| > R\}$ are open. $\{z \in \mathbb{C} : |z - z_0| \leq R\}$ is closed.

Example 1.2. $\mathbb{C}$ and the empty set $\varnothing$ are open. They are also closed!

Definition 1.3. The *boundary* of a set E, written ∂E, is the set of all boundary points of E. The *interior* of E is the set of all interior points of E. The *closure* of E, written $\overline{E}$, is the set of points in E together with all boundary points of E.

Example 1.3. If G is the open disk $\{z \in \mathbb{C} : |z - z_0| < R\}$ then

$$\overline{G} = \{z \in \mathbb{C} : |z - z_0| \leq R\} \qquad \text{and} \qquad \partial G = \{z \in \mathbb{C} : |z - z_0| = R\}.$$

That is, $\overline{G}$ is a closed disk and ∂G is a circle.

One notion that is somewhat subtle in the complex domain is the idea of *connectedness*. Intuitively, a set is connected if it is "in one piece." In the reals a set is connected if and only if it is an interval, so there is little reason to discuss the matter. However, in the plane there is a vast variety of connected subsets, so a definition is necessary.

Definition 1.4. Two sets $X, Y \subseteq \mathbb{C}$ are *separated* if there are disjoint open sets A and B so that $X \subseteq A$ and $Y \subseteq B$. A set $W \subseteq \mathbb{C}$ is *connected* if it is impossible to find two separated non-empty sets whose union is equal to W. A *region* is a connected open set.

The idea of separation is that the two open sets A and B ensure that X and Y cannot just "stick together." It is usually easy to check that a set is *not* connected. For example, the intervals $X = [0, 1)$ and $Y = (1, 2]$ on the real axis are separated: There are infinitely many choices for A and B that work; one choice is $A = D_1(0)$ (the open disk with center 0 and radius 1) and $B = D_1(2)$ (the open disk with center 2 and radius 1). Hence their union, which is $[0, 2] \setminus \{1\}$, is not connected. On the other hand, it is hard to use the definition to show that a set *is* connected, since we have to rule out any possible separation.

One type of connected set that we will use frequently is a curve.

Definition 1.5. A *path* or *curve* in $\mathbb{C}$ is the image of a continuous function $\gamma\colon [a,b] \to \mathbb{C}$, where $[a,b]$ is a closed interval in $\mathbb{R}$. The path γ is *smooth* if γ is differentiable.

We say that the curve is *parametrized* by γ. It is a customary and practical abuse of notation to use the same letter for the curve and its parametrization. We emphasize that a curve must have a parametrization, and that the parametrization must be defined and continuous on a *closed and bounded* interval $[a,b]$.

Since we may regard $\mathbb{C}$ as identified with $\mathbb{R}^2$, a path can be specified by giving two continuous real-valued functions of a real variable, $x(t)$ and $y(t)$, and setting $\gamma(t) = x(t) + y(t)i$. A curve is *closed* if $\gamma(a) = \gamma(b)$ and is a *simple closed curve* if $\gamma(s) = \gamma(t)$ implies $s = a$ and $t = b$ or $s = b$ and $t = a$, that is, the curve does not cross itself.

The following seems intuitively clear, but its proof requires more preparation in topology:

Proposition 1.5. *Any curve is connected.*

The next theorem gives an easy way to check whether an open set is connected, and also gives a very useful property of open connected sets.

Theorem 1.6. *If W is a subset of $\mathbb{C}$ that has the property that any two points in W can be connected by a curve in W then W is connected. On the other hand, if G is a connected open subset of $\mathbb{C}$ then any two points of G may be connected by a curve in G; in fact, we can connect any two points of G by a chain of horizontal and vertical segments lying in G.*

A *chain of segments* in G means the following: there are points z_0, z_1, $\ldots$, z_n so that, for each k, z_k and z_{k+1} are the endpoints of a horizontal or vertical segment which lies entirely in G. (It is not hard to parametrize such a chain, so it determines is a curve.)

As an example, let G be the open disk with center 0 and radius 2. Then any two points in G can be connected by a chain of at most 2 segments in G, so G is connected. Now let $G_0 = G \setminus \{0\}$; this is the punctured disk obtained by removing the center from G. Then G is open and it is connected, but now you may need more than two segments to connect points. For example, you need three segments to connect -1 to 1 since you cannot go through 0.

Warning: The second part of Theorem 1.6 is not generally true if G is not open. For example, circles are connected but there is no way to connect two distinct points of a circle by a chain of segments which are subsets of the circle. A more extreme example, discussed in topology texts, is the "topologist's sine curve," which is a connected set $S \subset \mathbb{C}$ that contains points that cannot be connected by a curve of any sort inside S.

The reader may skip the following proof. It is included to illustrate some common techniques in dealing with connected sets.

Proof of Theorem 1.6. Suppose, first, that any two points of G may be connected by a path that lies in G. If G is not connected then we can write it as a union of two non-empty separated subsets X and Y. So there are disjoint open sets A and B so that $X \subseteq A$ and $Y \subseteq B$. Since X and Y are disjoint we can find $a \in X$ and $b \in G$. Let γ be a path in G that connects a to b. Then $X_\gamma = X \cap \gamma$ and $Y_\gamma = Y \cap \gamma$ are disjoint and non-empty, their union is γ, and they are separated by A and B. But this means that γ is not connected, and this contradicts Proposition 1.5.

Now suppose that G is a connected open set. Choose a point $z_0 \in G$ and define two sets: A is the set of all points a so that there is a chain of segments in G connecting z_0 to a, and B is the set of points in G that are not in A.

Suppose a is in A. Since $a \in G$ there is an open disk D with center a that is contained in G. We can connect z_0 to any point z in D by following a chain of segments from z_0 to a, and then adding at most two segments in D that connect a to z. That is, each point of D is in A, so we have shown that A is open.

Now suppose b is in B. Since $b \in G$ there is an open disk D centered at b that lies in G. If z_0 could be connected to any point in D by a chain of segments in G then, extending this chain by at most two more segments, we could connect z_0 to b, and this is impossible. Hence z_0 cannot connect to any point of D by a chain of segments in G, so $D \subseteq B$. So we have shown that B is open.

Now G is the disjoint union of the two open sets A and B. If these are both non-empty then they form a separation of G, which is impossible. But z_0 is in A so A is not empty, and so B must be empty. That is, $G = A$, so z_0 can be connected to any point of G by a sequence of segments in G. Since z_0 could be any point in G, this finishes the proof. $\qquad\square$

1.4 Theorems from Calculus

Here are a few theorems from real calculus that we will make use of in the course of the text.

Theorem 1.7 (Extreme-Value Theorem). *Any continuous real-valued function defined on a closed and bounded subset of $\mathbb{R}^n$ has a minimum value and a maximum value.*

Theorem 1.8 (Mean-Value Theorem). *Suppose $I \subseteq \mathbb{R}$ is an interval, $f : I \to \mathbb{R}$ is differentiable, and $x, x + \Delta x \in I$. Then there is $0 < a < 1$ such that*

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x + a\Delta x) \, .$$

Many of the most important results of analysis concern combinations of limit operations. The most important of all calculus theorems combines differentiation and integration (in two ways):

Theorem 1.9 (Fundamental Theorem of Calculus). *Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then*

(a) *If F is defined by $F(x) = \int_a^x f(t)\,dt$ then F is differentiable and $F'(x) = f(x)$.*

(b) *If F is any antiderivative of f (that is, $F' = f$) then $\int_a^b f(x)\,dx = F(b) - F(a)$.*

For functions of several variables we can perform differentiation operations, or integration operations, in any order, if we have sufficient continuity:

Theorem 1.10 (Equality of mixed partials). *If the mixed partials $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are defined on an open set G and are continuous at a point (x_0, y_0) in G then they are equal at (x_0, y_0).*

Theorem 1.11 (Equality of iterated integrals). *If f is continuous on the rectangle given by $a \leq x \leq b$ and $c \leq y \leq d$ then the iterated integrals $\int_a^b \int_c^d f(x, y)\,dy\,dx$ and $\int_c^d \int_a^b f(x, y)\,dx\,dy$ are equal.*

Finally, we can apply differentiation and integration with respect to different variables in either order:

Theorem 1.12 (Leibniz's[4] Rule). *Suppose f is continuous on the rectangle R given by $a \leq x \leq b$ and $c \leq y \leq d$, and suppose the partial derivative $\frac{\partial f}{\partial x}$ exists and is continuous on R. Then*

$$\frac{d}{dx} \int_c^d f(x,y)\, dy = \int_c^d \frac{\partial f}{\partial x}(x,y)\, dy\,.$$

Exercises

1. Find the real and imaginary parts of each of the following:

    (a) $\frac{z-a}{z+a}$ $\quad (a \in \mathbb{R})$.

    (b) $\frac{3+5i}{7i+1}$.

    (c) $\left(\frac{-1+i\sqrt{3}}{2} \right)^3$.

    (d) i^n for any $n \in \mathbb{Z}$.

2. Find the absolute value and conjugate of each of the following:

    (a) $-2 + i$.

    (b) $(2+i)(4+3i)$.

    (c) $\frac{3-i}{\sqrt{2}+3i}$.

    (d) $(1+i)^6$.

3. Write in polar form:

    (a) $2i$.

    (b) $1 + i$.

    (c) $-3 + \sqrt{3}i$.

4. Write in rectangular form:

    (a) $\sqrt{2}\, e^{i3\pi/4}$.

    (b) $34\, e^{i\pi/2}$.

    (c) $-e^{i250\pi}$.

5. Find all solutions to the following equations:

    (a) $z^6 = 1$.

    (b) $z^4 = -16$.

    (c) $z^6 = -9$.

    (d) $z^6 - z^3 - 2 = 0$.

6. Show that

---

[4]Named after Gottfried Wilhelm Leibniz (1646–1716). For more information about Leibnitz, see `http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Leibnitz.html`.

(a) z is a real number if and only if $z = \bar{z}$;

(b) z is either real or purely imaginary if and only if $(\bar{z})^2 = z^2$.

7. Find all solutions of the equation $z^2 + 2z + (1 - i) = 0$.

8. Prove Theorem 1.1.

9. Show that if $z_1 z_2 = 0$ then $z_1 = 0$ or $z_2 = 0$.

10. Prove Lemma 1.2.

11. Use Lemma 1.2 to derive the triple angle formulas:

(a) $\cos 3\theta = \cos^3 \theta - 3\cos\theta \sin^2 \theta$.

(b) $\sin 3\theta = 3\cos^2 \theta \sin\theta - \sin^3 \theta$.

12. Prove Lemma 1.3.

13. Sketch the following sets in the complex plane:

(a) $\{z \in \mathbb{C} : |z - 1 + i| = 2\}$.

(b) $\{z \in \mathbb{C} : |z - 1 + i| \leq 2\}$.

(c) $\{z \in \mathbb{C} : \operatorname{Re}(z + 2 - 2i) = 3\}$.

(d) $\{z \in \mathbb{C} : |z - i| + |z + i| = 3\}$.

14. Suppose p is a polynomial with real coefficients. Prove that

(a) $\overline{p(z)} = p(\bar{z})$.

(b) $p(z) = 0$ if and only if $p(\bar{z}) = 0$.

15. Prove the reverse triangle inequality $|z_1 - z_2| \geq |z_1| - |z_2|$.

16. Use the previous exercise to show that $\left| \frac{1}{z^2 - 1} \right| \leq \frac{1}{3}$ for every z on the circle $z = 2e^{i\theta}$.

17. Sketch the following sets and determine whether they are open, closed, or neither; bounded; connected.

(a) $|z + 3| < 2$.

(b) $|\operatorname{Im} z| < 1$.

(c) $0 < |z - 1| < 2$.

(d) $|z - 1| + |z + 1| = 2$.

(e) $|z - 1| + |z + 1| < 3$.

18. What are the boundaries of the sets in the previous exercise?

19. The set E is the set of points z in $\mathbb{C}$ satisfying either z is real and $-2 < z < -1$, or $|z| < 1$, or $z = 1$ or $z = 2$.

    (a) Sketch the set E, being careful to indicate exactly the points that are in E.

    (b) Determine the interior points of E.

    (c) Determine the boundary points of E.

    (d) Determine the isolated points of E.

20. The set E in the previous exercise can be written in three different ways as the union of two disjoint nonempty separated subsets. Describe them, and in each case say briefly why the subsets are separated.

21. Let G be the annulus determined by the conditions $2 < |z| < 3$. This is a connected open set. Find the maximum number of horizontal and vertical segments in G needed to connect two points of G.

22. Prove Leibniz's Rule: Define $F(x) = \int_c^d f(x, y)\, dy$, get an expression for $F(x) - F(a)$ as an iterated integral by writing $f(x, y) - f(a, y)$ as the integral of $\frac{\partial f}{\partial x}$, interchange the order of integrations, and then differentiate using the Fundamental Theorem of Calculus.

Chapter 2

Differentiation

Mathematical study and research are very suggestive of mountaineering. Whymper made several efforts before he climbed the Matterhorn in the 1860's and even then it cost the life of four of his party. Now, however, any tourist can be hauled up for a small cost, and perhaps does not appreciate the difficulty of the original ascent. So in mathematics, it may be found hard to realise the great initial difficulty of making a little step which now seems so natural and obvious, and it may not be surprising if such a step has been found and lost again.
Louis Joel Mordell (1888–1972)

2.1 First Steps

A *(complex) function* f is a mapping from a subset $G \subseteq \mathbb{C}$ to $\mathbb{C}$ (in this situation we will write $f : G \to \mathbb{C}$ and call G the *domain* of f). This means that each element $z \in G$ gets mapped to exactly one complex number, called the *image* of z and usually denoted by $f(z)$. So far there is nothing that makes complex functions any more special than, say, functions from $\mathbb{R}^m$ to $\mathbb{R}^n$. In fact, we can construct many familiar looking functions from the standard calculus repertoire, such as $f(z) = z$ (the *identity map*), $f(z) = 2z + i$, $f(z) = z^3$, or $f(z) = \frac{1}{z}$. The former three could be defined on all of $\mathbb{C}$, whereas for the latter we have to exclude the origin $z = 0$. On the other hand, we could construct some functions which make use of a certain representation of z, for example, $f(x, y) = x - 2iy$, $f(x, y) = y^2 - ix$, or $f(r, \phi) = 2re^{i(\phi+\pi)}$.

Maybe *the* fundamental principle of analysis is that of a limit. The philosophy of the following definition is not restricted to complex functions, but for sake of simplicity we only state it for those functions.

Definition 2.1. Suppose f is a complex function with domain G and z_0 is an accumulation point of G. Suppose there is a complex number w_0 such that for every $\epsilon > 0$, we can find $\delta > 0$ so that for all $z \in G$ satisfying $0 < |z - z_0| < \delta$ we have $|f(z) - w_0| < \epsilon$. Then w_0 is the *limit* of f as z approaches z_0, in short

$$\lim_{z \to z_0} f(z) = w_0 \,.$$

This definition is the same as is found in most calculus texts. The reason we require that z_0 is an accumulation point of the domain is just that we need to be sure that there are points z of the domain which are arbitrarily close to z_0. Just as in the real case, the definition does not require

that z_0 is in the domain of f and, if z_0 is in the domain of f, the definition explicitly ignores the value of $f(z_0)$. That is why we require $0 < |z - z_0|$.

Just as in the real case the limit w_0 is unique if it exists. It is often useful to investigate limits by restricting the way the point z "approaches" z_0. The following is a easy consequence of the definition.

Lemma 2.1. *Suppose* $\lim_{z \to z_0} f(z)$ *exists and has the value* w_0, *as above. Suppose* $G_0 \subseteq G$, *and suppose* z_0 *is an accumulation point of* G_0. *If* f_0 *is the restriction of* f *to* G_0 *then* $\lim_{z \to z_0} f_0(z)$ *exists and has the value* w_0.

The definition of limit in the complex domain has to be treated with a little more care than its real companion; this is illustrated by the following example.

Example 2.1. $\lim_{z \to 0} \dfrac{\bar{z}}{z}$ does not exist.
To see this, we try to compute this "limit" as $z \to 0$ on the real and on the imaginary axis. In the first case, we can write $z = x \in \mathbb{R}$, and hence

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0} \frac{\bar{x}}{x} = \lim_{x \to 0} \frac{x}{x} = 1 \,.$$

In the second case, we write $z = iy$ where $y \in \mathbb{R}$, and then

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{y \to 0} \frac{\overline{iy}}{iy} = \lim_{y \to 0} \frac{-iy}{iy} = -1 \,.$$

So we get a different "limit" depending on the direction from which we approach 0. Lemma 2.1 then implies that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

On the other hand, the following "usual" limit rules are valid for complex functions; the proofs of these rules are everything but trivial and make for nice exercises.

Lemma 2.2. *Let* f *and* g *be complex functions and* $c, z_0 \in \mathbb{C}$.

(a) $\lim_{z \to z_0} f(z) + c \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (f(z) + c\,g(z))$

(b) $\lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (f(z) \cdot g(z))$

(c) $\lim_{z \to z_0} f(z) / \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (f(z)/g(z)) \,.$

In the last identity we have to make sure we do not divide by zero.

Because the definition of the limit is somewhat elaborate, the following fundamental definition looks almost trivial.

Definition 2.2. Suppose f is a complex function. If z_0 is in the domain of the function and either z_0 is an isolated point of the domain or

$$\lim_{z \to z_0} f(z) = f(z_0)$$

then f is *continuous* at z_0. More generally, f is *continuous* on $G \subseteq \mathbb{C}$ if f is continuous at every $z \in G$.

Just as in the real case, we can "take the limit inside" a continuous function:

Lemma 2.3. *If f is continuous at w_0 and $\lim_{z \to z_0} g(z) = w_0$ then $\lim_{z \to z_0} f(g(z)) = f(w_0)$. In other words,*

$$\lim_{z \to z_0} f(g(z)) = f\left(\lim_{z \to z_0} g(z)\right).$$

2.2 Differentiability and Analyticity

The fact that limits such as $\lim_{z \to 0} \frac{\bar{z}}{z}$ do not exist points to something special about complex numbers which has no parallel in the reals—we can express a function in a very compact way in one variable, yet it shows some peculiar behavior "in the limit." We will repeatedly notice this kind of behavior; one reason is that when trying to compute a limit of a function as, say, $z \to 0$, we have to allow z to approach the point 0 in any way. On the real line there are only two directions to approach 0—from the left or from the right (or some combination of those two). In the complex plane, we have an additional dimension to play with. This means that the statement "A complex function has a limit..." is in many senses stronger than the statement "A real function has a limit..." This difference becomes apparent most baldly when studying derivatives.

Definition 2.3. Suppose $f : G \to \mathbb{C}$ is a complex function and z_0 is an interior point of G. The *derivative of f at z_0* is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided this limit exists. In this case, f is called *differentiable* at z_0. If f is differentiable for all points in an open disk centered at z_0 then f is called *analytic* at z_0. The function f is *analytic* on the open set $G \subseteq \mathbb{C}$ if it is differentiable (and hence analytic) at every point in G. Functions which are differentiable (and hence analytic) in the whole complex plane $\mathbb{C}$ are called *entire*.

The difference quotient limit which defines $f'(z_0)$ can be rewritten as

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

This equivalent definition is sometimes easier to handle. Note that h is *not* a real number but can rather approach zero from anywhere in the complex plane.

The fact that the notions of differentiability and analyticity are actually different is seen in the following examples.

Example 2.2. The function $f(z) = z^3$ is entire, that is, analytic in $\mathbb{C}$: For any $z_0 \in \mathbb{C}$,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^3 - z_0^3}{z - z_0} = \lim_{z \to z_0} \frac{(z^2 + zz_0 + z_0^2)(z - z_0)}{z - z_0} = \lim_{z \to z_0} z^2 + zz_0 + z_0^2 = 3z_0^2.$$

Example 2.3. The function $f(z) = \bar{z}^2$ is differentiable at 0 and nowhere else (in particular, f is *not* analytic at 0): Let's write $z = z_0 + re^{i\phi}$. Then

$$\frac{\bar{z}^2 - \bar{z_0}^2}{z - z_0} = \frac{\left(\overline{z_0 + re^{i\phi}}\right)^2 - \bar{z_0}^2}{z_0 + re^{i\phi} - z_0} = \frac{\left(\bar{z_0} + re^{-i\phi}\right)^2 \bar{z_0}^2}{re^{i\phi}} = \frac{\bar{z_0}^2 + 2\bar{z_0}re^{-i\phi} + r^2 e^{-2i\phi} - \bar{z_0}^2}{re^{i\phi}}$$

$$= \frac{2\bar{z_0}re^{-i\phi} + r^2 e^{-2i\phi}}{re^{i\phi}} = 2\bar{z_0}e^{-2i\phi} + re^{-3i\phi}.$$

If $z_0 \neq 0$ then the limit of the right-hand side as $z \to z_0$ does not exist since $r \to 0$ and we get different answers for horizontal approach ($\phi = 0$) and for vertical approach ($\phi = \pi/2$). (A more entertaining way to see this is to use, for example, $z(t) = z_0 + \frac{1}{t}e^{it}$, which approaches z_0 as $t \to \infty$.) On the other hand, if $z_0 = 0$ then the right-hand side equals $re^{-3i\phi} = |z|e^{-3i\phi}$. Hence

$$\lim_{z \to 0} \left| \frac{\overline{z}^2}{z} \right| = \lim_{z \to 0} \left| |z|e^{-3i\phi} \right| = \lim_{z \to 0} |z| = 0 \,,$$

which implies that

$$\lim_{z \to 0} \frac{\overline{z}^2}{z} = 0 \,.$$

Example 2.4. The function $f(z) = \overline{z}$ is nowhere differentiable:

$$\lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0} = \lim_{z \to z_0} \frac{\overline{z - z_0}}{z - z_0} = \lim_{z \to 0} \frac{\overline{z}}{z}$$

does not exist, as discussed earlier.

The basic properties for derivatives are similar to those we know from real calculus. In fact, one should convince oneself that the following rules follow mostly from properties of the limit. (The 'chain rule' needs a little care to be worked out.)

Lemma 2.4. *Suppose f and g are differentiable at $z \in \mathbb{C}$, and that $c \in \mathbb{C}$, $n \in \mathbb{Z}$, and h is differentiable at $g(z)$.*

(a) $\left(f(z) + c\,g(z) \right)' = f'(z) + c\,g'(z)$

(b) $\left(f(z) \cdot g(z) \right)' = f'(z)g(z) + f(z)g'(z)$

(c) $\left(f(z)/g(z) \right)' = \dfrac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$

(d) $\left(z^n \right)' = nz^{n-1}$

(e) $\left(h(g(z)) \right)' = h'(g(z))g'(z) \,.$

In the third identity we have to be aware of division by zero.

We end this section with yet another differentiation rule, that for inverse functions. As in the real case, this rule is only defined for functions which are bijections. A function $f : G \to H$ is *one-to-one* if for every image $w \in H$ there is a unique $z \in G$ such that $f(z) = w$. The function is *onto* if every $w \in H$ has a preimage $z \in G$ (that is, there exists a $z \in G$ such that $f(z) = w$). A *bijection* is a function which is both one-to-one and onto. If $f : G \to H$ is a bijection then g is the *inverse of f* if for all $z \in H$, $f(g(z)) = z$.

Lemma 2.5. *Suppose G and H are open sets in $\mathbb{C}$, $f : G \to H$ is a bijection, $g : H \to G$ is the inverse function of f, and $z_0 \in H$. If f is differentiable at $g(z_0)$, $f'(g(z_0)) \neq 0$, and g is continuous at z_0 then g is differentiable at z_0 with*

$$g'(z_0) = \frac{1}{f'(g(z_0))} \,.$$

Proof. The function F defined by

$$F(z) = \begin{cases} \dfrac{f(w) - f(w_0)}{w - w_0} & \text{if } w \neq w_0, \\ f'(w_0) & \text{if } w = w_0 \end{cases}$$

is continuous at w_0. This appears when we calculate $g'(z_0)$:

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{f(g(z)) - f(g(z_0))} = \lim_{z \to z_0} \frac{1}{\dfrac{f(g(z)) - f(g(z_0))}{g(z) - g(z_0)}} = \lim_{z \to z_0} \frac{1}{F(g(z))} .$$

Now apply Lemma 2.3 to evaluate this last limit as

$$\frac{1}{F(g(z_0))} = \frac{1}{f'(g(z_0))} . \qquad \qquad \square$$

2.3 The Cauchy–Riemann Equations

Theorem 2.6. (a) *Suppose f is differentiable at $z_0 = x_0 + iy_0$. Then the partial derivatives of f satisfy*

$$\frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0) . \tag{2.1}$$

(b) *Suppose f is a complex function such that the partial derivatives f_x and f_y exist in an open disk centered at z_0 and are continuous at z_0. If these partial derivatives satisfy (2.1) then f is differentiable at z_0.*
In both cases (a) *and* (b)*, f' is given by*

$$f'(z_0) = \frac{\partial f}{\partial x}(z_0) .$$

Remarks. 1. It is traditional, and often convenient, to write the function f in terms of its real and imaginary parts. That is, we write $f(z) = f(x, y) = u(x, y) + iv(x, y)$ where u is the real part of f and v is the imaginary part. Then $f_x = u_x + iv_x$ and $-if_y = -i(u_y + iv_y) = v_y - iu_y$. Using this terminology we can rewrite the equation (2.1) equivalently as the following pair of equations:

$$\begin{aligned} u_x(x_0, y_0) &= v_y(x_0, y_0) \\ u_y(x_0, y_0) &= -v_x(x_0, y_0) . \end{aligned} \tag{2.2}$$

2. The partial differential equations (2.2) are called the *Cauchy–Riemann equations*, named after Augustin Louis Cauchy (1789–1857)[1] and Georg Friedrich Bernhard Riemann (1826–1866)[2].

3. As stated, (a) and (b) are not quite converse statements. However, we will later show that if f is *analytic* at $z_0 = x_0 + iy_0$ then u and v have continuous partials (of any order) at z_0. That is, later

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[1]For more information about Cauchy, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Cauchy.html`.
[2]For more information about Riemann, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Riemann.html`.

we will prove that $f = u + iv$ is analytic in an open set G if and only if u and v have continuous partials that satisfy (2.2) in G.

4. If u and v satisfy (2.2) and their second partials are also continuous then we obtain

$$u_{xx}(x_0, y_0) = v_{yx}(x_0, y_0) = v_{xy}(x_0, y_0) = -u_{yy}(x_0, y_0),$$

that is,

$$u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) = 0$$

and an analogous identity for v. Functions with continuous second partials satisfying this partial differential equation are called *harmonic*; we will study such functions in Chapter 6. Again, as we will see later, if f is analytic in an open set G then the partials of any order of u and v exist; hence we will show that the real and imaginary part of a function which is analytic on an open set are harmonic on that set.

Proof of Theorem 2.6. (a) If f is differentiable at $z_0 = (x_0, y_0)$ then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

As we saw in the last section we must get the same result if we restrict Δz to be on the real axis and if we restrict it to be on the imaginary axis. In the first case we have $\Delta z = \Delta x$ and

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = \frac{\partial f}{\partial x}(x_0, y_0).$$

In the second case we have $\Delta z = i\Delta y$ and

$$f'(z_0) = \lim_{i\Delta y \to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} = \lim_{\Delta y \to 0} \frac{1}{i} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = -i\frac{\partial f}{\partial y}(x_0, y_0)$$

(using $\frac{1}{i} = -i$). Thus we have shown that $f'(z_0) = f_x(z_0) = -if_y(z_0)$.

(b) To prove the statement in (b), "all we need to do" is prove that $f'(z_0) = f_x(z_0)$, assuming the Cauchy–Riemann equations and continuity of the partials. We first rearrange a difference quotient for $f'(z_0)$, writing $\Delta z = \Delta x + i\Delta y$:

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{f(z_0 + \Delta z) - f(z_0 + \Delta x) + f(z_0 + \Delta x) - f(z_0)}{\Delta z}$$

$$= \frac{f(z_0 + \Delta x + i\Delta y) - f(z_0 + \Delta x)}{\Delta z} + \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta z}$$

$$= \frac{\Delta y}{\Delta z} \cdot \frac{f(z_0 + \Delta x + i\Delta y) - f(z_0 + \Delta x)}{\Delta y} + \frac{\Delta x}{\Delta z} \cdot \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x}.$$

Now we rearrange $f_x(z_0)$:

$$f_x(z_0) = \frac{\Delta z}{\Delta z} \cdot f_x(z_0) = \frac{i\Delta y + \Delta x}{\Delta z} \cdot f_x(z_0) = \frac{\Delta y}{\Delta z} \cdot if_x(z_0) + \frac{\Delta x}{\Delta z} \cdot f_x(z_0)$$

$$= \frac{\Delta y}{\Delta z} \cdot f_y(z_0) + \frac{\Delta x}{\Delta z} \cdot f_x(z_0),$$

where we used equation (2.1) in the last step to convert if_x to $i(-if_y) = f_y$. Now we subtract our two rearrangements and take a limit:

$$
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - f_x(z_0)
$$

$$
= \lim_{\Delta z \to 0} \left[\frac{\Delta y}{\Delta z} \left(\frac{f(z_0 + \Delta x + i\Delta y) - f(z_0 + \Delta x)}{\Delta y} - f_y(z_0) \right) \right] \tag{2.3}
$$

$$
+ \lim_{\Delta z \to 0} \left[\frac{\Delta x}{\Delta z} \left(\frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} - f_x(z_0) \right) \right].
$$

We need to show that these limits are both 0. The fractions $\Delta x / \Delta z$ and $\Delta y / \Delta z$ are bounded by 1 in modulus so we just need to see that the limits of the expressions in parentheses are 0. The second term in (2.3) has a limit of 0 since, by definition,

$$
f_x(z_0) = \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x}
$$

and taking the limit as $\Delta z \to 0$ is the same as taking the limit as $\Delta x \to 0$. We can't do this for the first expression since both Δx and Δy are involved, and both change as $\Delta z \to 0$.

For the first term in (2.3) we apply Theorem 1.8, the real mean-value theorem, to the real and imaginary parts of f. This gives us real numbers a and b, with $0 < a, b < 1$, so that

$$
\frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0 + \Delta x, y_0)}{\Delta y} = u_y(x_0 + \Delta x, y_0 + a\Delta y)
$$

$$
\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0 + \Delta x, y_0)}{\Delta y} = v_y(x_0 + \Delta x, y_0 + b\Delta y).
$$

Using these expressions, we have

$$
\frac{f(z_0 + \Delta x + i\Delta y) - f(z_0 + \Delta x)}{\Delta y} - f_y(z_0)
$$

$$
= u_y(x_0 + \Delta x, y_0 + a\Delta y) + iv_y(x_0 + \Delta x, y_0 + b\Delta y) - (u_y(x_0, y_0) + iv_y(x_0, y_0))
$$

$$
= (u_y(x_0 + \Delta x, y_0 + a\Delta y) - u_y(x_0, y_0)) + i\,(v_y(x_0 + \Delta x, y_0 + a\Delta y) - v_y(x_0, y_0)).
$$

Finally, the two differences in parentheses have zero limit as $\Delta z \to 0$ because u_y and v_y are continuous at (x_0, y_0). $\qquad \square$

2.4 Constants and Connectivity

One of the first applications of the mean-value theorem in real calculus is to show that if a function has zero derivative everywhere on an interval then it must be constant. The proof is very easy: The mean-value theorem for a real function says $f(x + \Delta x) - f(x) = f'(x + a\Delta x)\Delta x$ where $0 < a < 1$. If we know that f' is always zero then we know that $f'(x + a\Delta x) = 0$, so $f(x + \Delta x) = f(x)$. This says that all values of f must be the same, so f is a constant.

However the mean-value theorem does not have a simple analog for complex valued functions, so we need another argument to prove that functions with derivative that are always 0 must be

constant. In fact, this isn't really true. For example, if the domain of f consists of all complex numbers with non-zero real part and

$$f(z) = \begin{cases} 1 & \text{if } \operatorname{Re} z > 0, \\ -1 & \text{if } \operatorname{Re} z < 0, \end{cases}$$

then $f'(z) = 0$ for all z in the domain of f but f is not constant.

This may seem like a silly example, but it illustrates an important fact about complex functions. In many cases during the course we will want to conclude that a function is constant, and in each case we will have to allow for examples like the above. The fundamental problem is that the domain in this example is not connected, and in fact the correct theorem is:

Theorem 2.7. *If the domain of f is a region $G \subseteq \mathbb{C}$ and $f'(z) = 0$ for all z in G then f is a constant.*

Proof. First, suppose that H is a horizontal line segment in G. Consider the real part $u(z)$ for $z \in H$. Since H is a horizontal segment, y is constant on H, so we can consider $u(z)$ to be just a function of x. But $u_x(z) = \operatorname{Re}(f'(z)) = 0$ so, by the real version of the theorem, $u(z)$ is constant on this horizontal segment. We can argue the same way to see that the imaginary part $v(z)$ of $f(z)$ is constant on H, since $v_x(z) = \operatorname{Im}(f'(z)) = 0$. Since both the real and imaginary parts of f are constant on H, f itself is constant on H.

Next, suppose that V is a vertical segment that is contained in G, and consider the real part $u(z)$ for z on V. As above, we can consider $u(z)$ to be just a function of y and, using the Cauchy–Riemann equations, $u_y(z) = -v_x(z) = -\operatorname{Im}(f'(z)) = 0$. Thus u is constant on V, and similarly v is constant on V, so f is constant on V.

Now we can prove the theorem using these two facts: Fix a starting point z_0 in G and let $b = f(z_0)$. Connect z_0 to a point z_1 by a horizontal segment H in G; then f is constant on H so $f(z_1) = f(z_0) = b$. Now connect z_1 to a point z_2 by a vertical segment V in G; then f is constant on V so $f(z_2) = f(z_1) = b$. Now connect z_2 to a point z_3 by a horizontal segment and conclude that $f(z_3) = b$. Repeating this argument we see that $f(z) = b$ for all points that can be connected to z_0 in this way by a finite sequence of horizontal and vertical segments. Theorem 1.6 says that this is always possible. $\square$

There are a number of surprising applications of this theorem; see Exercises 13 and 14 for a start.

Exercises

1. Use the definition of limit to show that $\lim_{z \to z_0} (az + b) = az_0 + b$.

2. Evaluate the following limits or explain why they don't exist.

    (a) $\lim\limits_{z \to i} \frac{iz^3 - 1}{z + i}$.

    (b) $\lim\limits_{z \to 1 - i} x + i(2x + y)$.

3. Prove Lemma 2.2.

4. Prove Lemma 2.2 by using the formula for f' given in Theorem 2.6.

5. Apply the definition of the derivative to give a direct proof that $f'(z) = -\frac{1}{z^2}$ when $f(z) = \frac{1}{z}$.

6. Show that if f is differentiable at z then f is continuous at z.

7. Prove Lemma 2.3.

8. Prove Lemma 2.4.

9. If $u(x, y)$ and $v(x, y)$ are continuous (respectively differentiable) does it follow that $f(z) = u(x, y) + iv(x, y)$ is continuous (resp. differentiable)? If not, provide a counterexample.

10. Where are the following functions differentiable? Where are they analytic? Determine their derivatives at points where they are differentiable.

    (a) $f(z) = e^{-x}e^{-iy}$.
    (b) $f(z) = 2x + ixy^2$.
    (c) $f(z) = x^2 + iy^2$.
    (d) $f(z) = e^x e^{-iy}$.
    (e) $f(z) = \cos x \cosh y - i \sin x \sinh y$.
    (f) $f(z) = \operatorname{Im} z$.
    (g) $f(z) = |z|^2 = x^2 + y^2$.
    (h) $f(z) = z \operatorname{Im} z$.
    (i) $f(z) = \frac{ix+1}{y}$.
    (j) $f(z) = 4(\operatorname{Re} z)(\operatorname{Im} z) - i(\bar{z})^2$.
    (k) $f(z) = 2xy - i(x + y)^2$.
    (l) $f(z) = z^2 - \bar{z}^2$.

11. Prove that if $f(z)$ is given by a polynomial in z then f is entire. What can you say if $f(z)$ is given by a polynomial in $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$?

12. Consider the function
$$f(z) = \begin{cases} \dfrac{xy(x + iy)}{x^2 + y^2} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

    (As always, $z = x + iy$.) Show that f satisfies the Cauchy–Riemann equations at the origin $z = 0$, yet f is not differentiable at the origin. Why doesn't this contradict Theorem 2.6 (b)?

13. Prove: If f is analytic in the region $G \subseteq \mathbb{C}$ and always real valued, then f is constant in G. (*Hint*: Use the Cauchy–Riemann equations to show that $f' = 0$.)

14. Prove: If $f(z)$ and $\overline{f(z)}$ are both analytic in the region $G \subseteq \mathbb{C}$ then $f(z)$ is constant in G.

15. Suppose $f(z)$ is entire, with real and imaginary parts $u(z)$ and $v(z)$ satisfying $u(z)v(z) = 3$ for all z. Show that f is constant.

16. Is $\frac{x}{x^2+y^2}$ harmonic? What about $\frac{x^2}{x^2+y^2}$?

17. The general real homogeneous quadratic function of (x, y) is

$$u(x, y) = ax^2 + bxy + cy^2,$$

where a, b and c are real constants.

(a) Show that u is harmonic if and only if $a = -c$.

(b) If u is harmonic then show that it is the real part of a function of the form $f(z) = Az^2$, where A is a complex constant. Give a formula for A in terms of the constants a, b and c.

Chapter 3

Examples of Functions

> *Obvious is the most dangerous word in mathematics.*
> E. T. Bell

3.1 Möbius Transformations

The first class of functions that we will discuss in some detail are built from linear polynomials.

Definition 3.1. A *linear fractional transformation* is a function of the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$. If $ad - bc \neq 0$ then f is called a *Möbius*[1] *transformation.*

Exercise 11 of the previous chapter states that any polynomial (in z) is an entire function. From this fact we can conclude that a linear fractional transformation $f(z) = \frac{az+b}{cz+d}$ is analytic in $\mathbb{C} \setminus \left\{ -\frac{d}{c} \right\}$ (unless $c = 0$, in which case f is entire).

One property of Möbius transformations, which is quite special for complex functions, is the following.

Lemma 3.1. *Möbius transformations are bijections. In fact, if $f(z) = \frac{az+b}{cz+d}$ then the inverse function of f is given by*

$$f^{-1}(z) = \frac{dz - b}{-cz + a}.$$

Remark. Notice that the inverse of a Möbius transformation is another Möbius transformation.

Proof. Note that $f : \mathbb{C} \setminus \{ -\frac{d}{c} \} \to \mathbb{C} \setminus \{ \frac{a}{c} \}$. Suppose $f(z_1) = f(z_2)$, that is,

$$\frac{az_1 + b}{cz_1 + d} = \frac{az_2 + b}{cz_2 + d}.$$

This is equivalent (unless the denominators are zero) to

$$(az_1 + b)(cz_2 + d) = (az_2 + b)(cz_1 + d),$$

---

[1] Named after August Ferdinand Möbius (1790–1868). For more information about Möbius, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Mobius.html`.

which can be rearranged to

$$(ad - bc)(z_1 - z_2) = 0\,.$$

Since $ad - bc \neq 0$ this implies that $z_1 = z_2$, which means that f is one-to-one. The formula for $f^{-1} : \mathbb{C} \setminus \{\frac{a}{c}\} \to \mathbb{C} \setminus \{-\frac{d}{c}\}$ can be checked easily. Just like f, f^{-1} is one-to-one, which implies that f is onto. $\qquad \square$

Aside from being prime examples of one-to-one functions, Möbius transformations possess fascinating geometric properties. En route to an example of such, we introduce some terminology. Special cases of Möbius transformations are *translations* $f(z) = z + b$, *dilations* $f(z) = az$, and *inversions* $f(z) = \frac{1}{z}$. The next result says that if we understand those three special transformations, we understand them all.

Proposition 3.2. *Suppose $f(z) = \frac{az+b}{cz+d}$ is a linear fractional transformation. If $c = 0$ then*

$$f(z) = \frac{a}{d}z + \frac{b}{d}\,,$$

if $c \neq 0$ then

$$f(z) = \frac{bc - ad}{c^2}\frac{1}{z + \frac{d}{c}} + \frac{a}{c}\,.$$

In particular, every linear fractional transformation is a composition of translations, dilations, and inversions.

Proof. Simplify. $\qquad \square$

With the last result at hand, we can tackle the promised theorem about the following geometric property of Möbius transformations.

Theorem 3.3. *Möbius transformations map circles and lines into circles and lines.*

Proof. Translations and dilations certainly map circles and lines into circles and lines, so by the last proposition, we only have to prove the theorem for the inversion $f(z) = \frac{1}{z}$.

Before going on we find a standard form for the equation of a straight line. Starting with $ax + by = c$ (where $z = x + iy$), let $\alpha = a + bi$. Then $\overline{\alpha}z = ax + by + i(ay - bx)$ so $\overline{\alpha}z + \alpha\overline{z} = \overline{\alpha}z + \overline{\overline{\alpha}z} = 2\operatorname{Re}(\overline{\alpha}z) = 2ax + 2by$. Hence our standard equation for a line becomes

$$\overline{\alpha}z + \alpha\overline{z} = 2c, \quad \text{or} \quad \operatorname{Re}(\overline{\alpha}z) = c. \tag{3.1}$$

First case: Given a circle centered at z_0 with radius r, we can modify its defining equation $|z - z_0| = r$ as follows:

$$|z - z_0|^2 = r^2$$
$$(z - z_0)\overline{(z - z_0)} = r^2$$
$$z\overline{z} - z_0\overline{z} - z\overline{z_0} + z_0\overline{z_0} = r^2$$
$$|z|^2 - z_0\overline{z} - z\overline{z_0} + |z_0|^2 - r^2 = 0\,.$$

Now we want to transform this into an equation in terms of w, where $w = \frac{1}{z}$. If we solve $w = \frac{1}{z}$ for z we get $z = \frac{1}{w}$, so we make this substitution in our equation:

$$\left|\frac{1}{w}\right|^2 - z_0\frac{1}{\overline{w}} - \overline{z_0}\frac{1}{w} + |z_0|^2 - r^2 = 0$$

$$1 - z_0 w - \overline{z_0 w} + |w|^2\left(|z_0|^2 - r^2\right) = 0.$$

(To get the second line we multiply by $|w|^2 = w\overline{w}$ and simplify.) Now if r happens to be equal to $|z_0|^2$ then this equation becomes $1 - z_0 w - \overline{z_0 w} = 0$, which is of the form (3.1) with $\alpha = \overline{z_0}$, so we have a straight line in terms of w. Otherwise $|z_0|^2 - r^2$ is non-zero so we can divide our equation by it. We obtain

$$|w|^2 - \frac{z_0}{|z_0|^2 - r^2}w - \frac{\overline{z_0}}{|z_0|^2 - r^2}\overline{w} + \frac{1}{|z_0|^2 - r^2} = 0.$$

We define

$$w_0 = \frac{\overline{z_0}}{|z_0|^2 - r^2}, \qquad s^2 = |w_0|^2 - \frac{1}{|z_0|^2 - r^2} = \frac{|z_0|^2}{(|z_0|^2 - r^2)^2} - \frac{|z_0|^2 - r^2}{(|z_0|^2 - r^2)^2} = \frac{r^2}{(|z_0|^2 - r^2)^2}.$$

Then we can rewrite our equation as

$$|w|^2 - \overline{w_0}w - w_0\overline{w} + |w_0|^2 - s^2 = 0$$

$$w\overline{w} - w_0\overline{w} - w\overline{w_0} + w_0\overline{w_0} = s^2$$

$$(w - w_0)\overline{(w - w_0)} = s^2$$

$$|w - w_0|^2 = s^2.$$

This is the equation of a circle in terms of w, with center w_0 and radius s.

Second case: We start with the equation of a line in the form (3.1) and rewrite it in terms of w, as above, by substituting $z = \frac{1}{w}$ and simplifying. We get

$$\overline{z_0}\overline{w} + z_0 w = 2cw\overline{w}.$$

If $c = 0$, this describes a line in the form (3.1) in terms of w. Otherwise we can divide by $2c$:

$$w\overline{w} - \frac{\overline{z_0}}{2c}\overline{w} - \frac{z_0}{2c}w = 0$$

$$\left(w - \frac{\overline{z_0}}{2c}\right)\left(\overline{w} - \frac{z_0}{2c}\right) - \frac{|z_0|^2}{4c^2} = 0$$

$$\left|w - \frac{\overline{z_0}}{2c}\right|^2 = \frac{|z_0|^2}{4c^2}.$$

This is the equation of a circle with center $\frac{\overline{z_0}}{2c}$ and radius $\frac{|z_0|}{2|c|}$. $\qquad\square$

There is one fact about Möbius transformations that is very helpful to understanding their geometry. In fact, it is much more generally useful:

Lemma 3.4. *Suppose f is analytic at a with $f'(a) \neq 0$ and suppose γ_1 and γ_2 are two smooth curves which pass through a, making an angle of θ with each other. Then f transforms γ_1 and γ_2 into smooth curves which meet at $f(a)$, and the transformed curves make an angle of θ with each other.*

In brief, an analytic function with non-zero derivative *preserves angles*. Functions which preserve angles in this way are also called *conformal*.

Proof. For $k = 1, 2$ we write γ_k parametrically, as $z_k(t) = x_k(t) + iy_k(t)$, so that $z_k(0) = a$. The complex number $z'_k(0)$, considered as a vector, is the tangent vector to γ_k at the point a. Then f transforms the curve γ_k to the curve $f(\gamma_k)$, parameterized as $f(z_k(t))$. If we differentiate $f(z_k(t))$ at $t = 0$ and use the chain rule we see that the tangent vector to the transformed curve at the point $f(a)$ is $f'(a)z'_k(0)$. Since $f'(a) \neq 0$ the transformation from $z'_1(0)$ and $z'_2(0)$ to $f'(a)z'_1(0)$ and $f'(a)z'_2(0)$ is a dilation. A dilation is the composition of a scale change and a rotation and both of these preserve the angles between vectors. $\square$

3.2 Infinity and the Cross Ratio

Infinity is not a number—this is true whether we use the complex numbers or stay in the reals. However, for many purposes we can work with infinity in the complexes much more naturally and simply than in the reals.

In the complex sense there is only one infinity, written ∞. In the real sense there is also a "negative infinity", but $-\infty = \infty$ in the complex sense. In order to deal correctly with infinity we have to realize that we are always talking about a limit, and complex numbers have infinite limits if they can become larger in magnitude than any preassigned limit. For completeness we repeat the usual definitions:

Definition 3.2. Suppose G is a set of complex numbers and f is a function from G to $\mathbb{C}$.

(a) $\lim_{z \to z_0} f(z) = \infty$ means that for every $M > 0$ we can find $\delta > 0$ so that, for all $z \in G$ satisfying $0 < |z - z_0| < \delta$, we have $|f(z)| > M$.

(b) $\lim_{z \to \infty} f(z) = L$ means that for every $\epsilon > 0$ we can find $N > 0$ so that, for all $z \in G$ satisfying $|z| > N$, we have $|f(z) - L| < \epsilon$.

(c) $\lim_{z \to \infty} f(z) = \infty$ means that for every $M > 0$ we can find $N > 0$ so that, for all $z \in G$ satisfying $|z| > N$ we have $|f(z)| > M$.

In the first definition we require that z_0 is an accumulation point of G while in the second and third we require that ∞ is an "extended accumulation point" of G, in the sense that for every $B > 0$ there is some $z \in G$ with $|z| > B$.

The usual rules for working with infinite limits are still valid in the complex numbers. In fact, it is a good idea to make infinity an honorary complex number so that we can more easily manipulate infinite limits. We do this by defining a new set, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In this new set we define algebraic rules for dealing with infinity based on the usual laws of limits. For example, if $\lim_{z \to z_0} f(z) = \infty$ and $\lim_{z \to z_0} g(z) = a$ is finite then the usual "limit of sum = sum of limits" rule gives $\lim_{z \to z_0} (f(z) + g(z)) = \infty$. This leads to the addition rule $\infty + a = \infty$. We summarize these rules:

Definition 3.3. Suppose $a \in \mathbb{C}$.

(a) $\infty + a = a + \infty = \infty$

(b) $\infty \cdot a = a \cdot \infty = \infty \cdot \infty = \infty$ if $a \neq 0$.

(c) $\dfrac{a}{\infty} = 0$ and $\dfrac{a}{0} = \infty$ if $a \neq 0$.

If a calculation involving infinity is not covered by the rules above then we must investigate the limit more carefully. For example, it may seem strange that $\infty + \infty$ is not defined, but if we take the limit of $z + (-z) = 0$ as $z \to \infty$ we will get 0, but the individual limits of z and $-z$ are both ∞.

Now we reconsider Möbius transformations with infinity in mind. For example, $f(z) = \frac{1}{z}$ is now defined for $z = 0$ and $z = \infty$, with $f(0) = \infty$ and $f(\infty) = 0$, so the proper domain for $f(z)$ is actually $\hat{\mathbb{C}}$. Let's consider the other basic types of Möbius transformations. A translation $f(z) = z + b$ is now defined for $z = \infty$, with $f(\infty) = \infty + b = \infty$, and a dilation $f(z) = az$ (with $a \neq 0$) is also defined for $z = \infty$, with $f(\infty) = a \cdot \infty = \infty$. Since every Möbius transformation can be expressed as a composition of translations, dilations and the inversion $f(z) = \frac{1}{z}$ we see that every Möbius transformation may be interpreted as a transformation of $\hat{\mathbb{C}}$ onto $\hat{\mathbb{C}}$. The general case is summarized below:

Lemma 3.5. *Let f be the Möbius transformation*

$$f(z) = \frac{az + b}{cz + d}.$$

Then f is defined for all $z \in \hat{\mathbb{C}}$. If $c = 0$ then $f(\infty) = \infty$, and, otherwise,

$$f(\infty) = \frac{a}{c} \qquad and \qquad f\left(-\frac{d}{c}\right) = \infty.$$

With this interpretation in mind we can add some insight to Theorem 3.3. Recall that $f(z) = \frac{1}{z}$ transforms circles that pass through the origin to straight lines, but the point $z = 0$ must be excluded from the circle. However, now we can put it back, so f transforms circles that pass through the origin to straight lines *plus* ∞. If we remember that ∞ corresponds to being arbitrarily far away from the origin we can visualize a line plus infinity as a circle passing through ∞. If we make this a definition then Theorem 3.3 can be expressed very simply: any Möbius transformation of $\hat{\mathbb{C}}$ transforms circles to circles. For example, the transformation

$$f(z) = \frac{z + i}{z - i}$$

transforms $-i$ to 0, i to ∞, and 1 to i. The three points $-i$, i and 1 determine a circle—the unit circle $|z| = 1$—and the three image points 0, ∞ and i also determine a circle—the imaginary axis plus the point at infinity. Hence f transforms the unit circle onto the imaginary axis plus the point at infinity.

This example relied on the idea that three distinct points in $\hat{\mathbb{C}}$ determine uniquely a circle passing through them. If the three points are on a straight line or if one of the points is ∞ then the circle is a straight line plus ∞. Conversely, if we know where three distinct points in $\hat{\mathbb{C}}$ are transformed by a Möbius transformation then we should be able to figure out everything about the transformation. There is a computational device that makes this easier to see.

Definition 3.4. If z, z_1, z_2, and z_3 are any four points in $\hat{\mathbb{C}}$ with z_1, z_2, and z_3 distinct, then their *cross-ratio* is defined by

$$[z, z_1, z_2, z_3] = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

Here if $z = z_3$, the result is infinity, and if one of z, z_1, z_2, or z_3 is infinity, then the two terms on the right containing it are canceled.

Lemma 3.6. *If f is defined by $f(z) = [z, z_1, z_2, z_3]$ then f is a Möbius transformation which satisfies*

$$f(z_1) = 0, \quad f(z_2) = 1, \quad f(z_3) = \infty.$$

Moreover, if g is any Möbius transformation which transforms z_1, z_2 and z_3 as above then $g(z) = f(z)$ for all z.

Proof. Everything should be clear except the final uniqueness statement. By Lemma 3.1 the inverse f^{-1} is a Möbius transformation and, by Exercise 7 in this chapter, the composition $h = g \circ f^{-1}$ is a Möbius transformation. Notice that $h(0) = g(f^{-1}(0)) = g(z_1) = 0$. Similarly, $h(1) = 1$ and $h(\infty) = \infty$. If we write $h(z) = \frac{az+b}{cz+d}$ then

$$0 = h(0) = \frac{b}{d} \implies b = 0$$

$$\infty = h(\infty) = \frac{a}{c} \implies c = 0$$

$$1 = h(1) = \frac{a+b}{c+d} = \frac{a+0}{0+d} = \frac{a}{d} \implies a = d,$$

so $h(z) = \frac{az+b}{cz+d} = \frac{az+0}{0+d} = \frac{a}{d}z = z$. But since $h(z) = z$ for all z we have $h(f(z)) = f(z)$ and so $g(z) = g \circ (f^{-1} \circ f)(z) = (g \circ f^{-1}) \circ f(z) = h(f(z)) = f(z)$. $\qquad\square$

So if we want to map three given points of $\hat{\mathbb{C}}$ to 0, 1 and ∞ by a Möbius transformation then the cross-ratio gives us the only way to do it. What if we have three points z_1, z_2 and z_3 and we want to map them to three other points, w_1, w_2 and w_3?

Theorem 3.7. *Suppose z_1, z_2 and z_3 are distinct points in $\hat{\mathbb{C}}$ and w_1, w_2 and w_3 are distinct points in $\hat{\mathbb{C}}$. Then there is a unique Möbius transformation h satisfying $h(z_1) = w_1$, $h(z_2) = w_2$ and $h(z_3) = w_3$.*

Proof. Let $h = g^{-1} \circ f$ where $f(z) = [z, z_1, z_2, z_3]$ and $g(w) = [w, w_1, w_2, w_3]$. Uniqueness follows as in the proof of Lemma 3.6. $\qquad\square$

This theorem gives an explicit way to determine h from the points z_j and w_j but, in practice, it is often easier to determine h directly from the conditions $f(z_k) = w_k$ (by solving for a, b, c and d).

3.3   Exponential and Trigonometric Functions

To define the complex exponential function, we once more borrow concepts from calculus, namely the real exponential function[2] and the real sine and cosine, and—in addition—finally make sense of the notation $e^{it} = \cos t + i \sin t$.

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[2]It is a nontrivial question how to define the real exponential function. Our preferred way to do this is through a power series: $e^x = \sum_{k \geq 0} x^k / k!$. In light of this definition, the reader might think we should have simply defined the

Definition 3.5. The *(complex) exponential function* is defined for $z = x + iy$ as

$$\exp(z) = e^x \left(\cos y + i \sin y\right) = e^x e^{iy}.$$

This definition seems a bit arbitrary, to say the least. Its first justification is that all exponential rules which we are used to from real numbers carry over to the complex case. They mainly follow from Lemma 1.2 and are collected in the following.

Lemma 3.8. *For all* $z, z_1, z_2 \in \mathbb{C}$,

(a) $\exp(z_1) \exp(z_2) = \exp(z_1 + z_2)$

(b) $\frac{1}{\exp(z)} = \exp(-z)$

(c) $\exp(z + 2\pi i) = \exp(z)$

(d) $|\exp(z)| = \exp(\operatorname{Re} z)$

(e) $\exp(z) \neq 0$

(f) $\frac{d}{dz} \exp(z) = \exp(z).$

Remarks. 1. The third identity is a very special one and has no counterpart for the real exponential function. It says that the complex exponential function is *periodic* with period $2\pi i$. This has many interesting consequences; one that may not seem too pleasant at first sight is the fact that the complex exponential function is not one-to-one.

2. The last identity is not only remarkable, but we invite the reader to meditate on its proof. When proving this identity through the Cauchy–Riemann equations for the exponential function, one can get another strong reason why Definition 3.5 is reasonable. Finally, note that the last identity also says that exp is entire.

We should make sure that the complex exponential function specializes to the real exponential function for real arguments: if $z = x \in \mathbb{R}$ then

$$\exp(x) = e^x \left(\cos 0 + i \sin 0\right) = e^x.$$

The trigonometric functions—sine, cosine, tangent, cotangent, etc.—have their complex analogues, however, they don't play the same prominent role as in the real case. In fact, we can define them as merely being special combinations of the exponential function.

Definition 3.6. The *(complex) sine* and *cosine* are defined as

$$\sin z = \frac{1}{2i} \left(\exp(iz) - \exp(-iz)\right) \qquad \text{and} \qquad \cos z = \frac{1}{2} \left(\exp(iz) + \exp(-iz)\right),$$

respectively. The *tangent* and *cotangent* are defined as

$$\tan z = \frac{\sin z}{\cos z} = -i \frac{\exp(2iz) - 1}{\exp(2iz) + 1} \qquad \text{and} \qquad \cot z = \frac{\cos z}{\sin z} = i \frac{\exp(2iz) + 1}{\exp(2iz) - 1},$$

respectively.

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complex exponential function through a complex power series. In fact, this is possible (and an elegant definition); however, one of the promises of these lecture notes is to introduce complex power series as late as possible. We agree with those readers who think that we are "cheating" at this point, as we borrow the concept of a (real) power series to define the real exponential function.
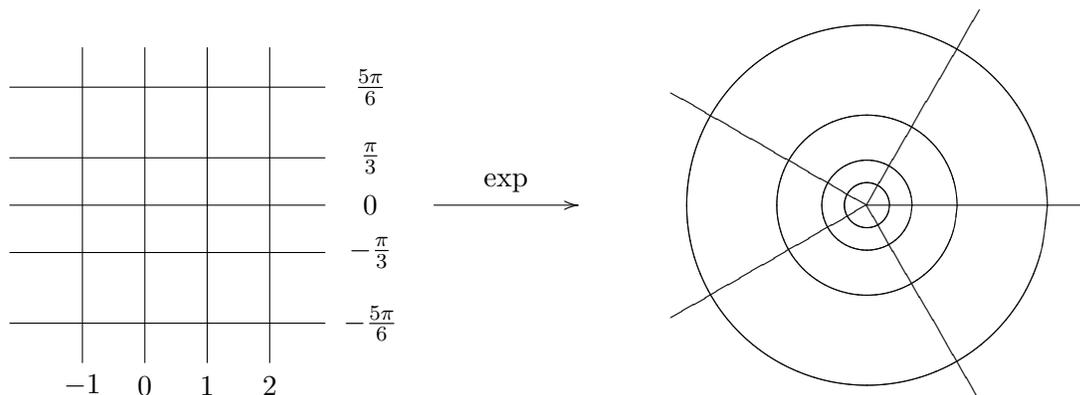
Figure 3.1: Image properties of the exponential function.

Note that to write tangent and cotangent in terms of the exponential function, we used the fact that $\exp(z)\exp(-z) = \exp(0) = 1$. Because exp is entire, so are sin and cos.

As with the exponential function, we should first make sure that we're not redefining the real sine and cosine: if $z = x \in \mathbb{R}$ then

$$\sin x = \frac{1}{2i}\left(\exp(ix) - \exp(-ix)\right) = \frac{1}{2i}\left(\cos x + i\sin x - (\cos(-x) + i\sin(-x))\right) = \sin x\,.$$

(The 'sin' on the left denotes the complex sine, the one on the right the real sine.) A similar calculation holds for the cosine. Not too surprising, the following properties follow mostly from Lemma 3.8.

Lemma 3.9. *For all* $z, z_1, z_2 \in \mathbb{C}$,

$$\sin(-z) = -\sin z \qquad\qquad \cos(-z) = \cos z$$
$$\sin(z + 2\pi) = \sin z \qquad\qquad \cos(z + 2\pi) = \cos z$$
$$\tan(z + \pi) = \tan z \qquad\qquad \cot(z + \pi) = \cot z$$
$$\sin(z + \pi/2) = \cos z \qquad\qquad \cos(z + \pi/2) = -\sin z$$
$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \qquad\qquad \cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$
$$\cos^2 z + \sin^2 z = 1 \qquad\qquad \cos^2 z - \sin^2 z = \cos(2z)$$
$$\sin' z = \cos z \qquad\qquad \cos' z = -\sin z\,.$$

Finally, one word of caution: unlike in the real case, the complex sine and cosine are *not* bounded—consider, for example, $\sin(iy)$ as $y \to \pm\infty$.

We end this section with a remark on hyperbolic trig functions. The hyperbolic sine, cosine, tangent, and cotangent are defined as in the real case:

$$\sinh z = \frac{1}{2}\left(\exp(z) - \exp(-z)\right) \qquad\qquad \cosh z = \frac{1}{2}\left(\exp(z) + \exp(-z)\right)$$
$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{\exp(2z) - 1}{\exp(2z) + 1} \qquad\qquad \coth z = \frac{\cosh z}{\sinh z} = \frac{\exp(2z) + 1}{\exp(2z) - 1}\,.$$

As such, they are not only yet more special combinations of the exponential function, but they are also related with the trigonometric functions via

$$\sinh(iz) = i \sin z \qquad \text{and} \qquad \cosh(iz) = \cos z \,.$$

3.4 The Logarithm and Complex Exponentials

The complex logarithm is the first function we'll encounter that is of a somewhat tricky nature. It is motivated as being the inverse function to the exponential function, that is, we're looking for a function $\mathcal{L}$og such that

$$\exp(\mathcal{L}\text{og}\, z) = z = \mathcal{L}\text{og}(\exp z) \,.$$

As we will see shortly, this is too much to hope for. Let's write, as usual, $z = r\, e^{i\phi}$, and suppose that $\mathcal{L}$og $z = u(z) + iv(z)$. Then for the first equation to hold, we need

$$\exp(\mathcal{L}\text{og}\, z) = e^u e^{iv} = r\, e^{i\phi} = z \,,$$

that is, $e^u = r = |z| \iff u = \ln|z|$ (where ln denotes the real natural logarithm; in particular we need to demand that $z \neq 0$), and $e^{iv} = e^{i\phi} \iff v = \phi + 2\pi k$ for some $k \in \mathbb{Z}$. A reasonable definition of a logarithm function $\mathcal{L}$og would hence be to set $\mathcal{L}$og $z = \ln|z| + i\,\mathcal{A}\text{rg}\, z$ where $\mathcal{A}\text{rg}\, z$ gives the argument for the complex number z according to some convention—for example, we could agree that the argument is always in $(-\pi, \pi]$, or in $[0, 2\pi)$, etc. The problem is that we need to stick to this convention. On the other hand, as we saw, we could just use a different argument convention and get another reasonable 'logarithm.' Even worse, by defining the multi-valued map

$$\arg z = \{\phi : \phi \text{ is a possible argument of } z\}$$

and defining the multi-valued logarithm as

$$\log z = \ln|z| + i \arg z \,,$$

we get something that's not a function, yet it satisfies

$$\exp(\log z) = z \,.$$

We invite the reader to check this thoroughly; in particular, one should note how the periodicity of the exponential function takes care of the multi-valuedness of our 'logarithm' log.

log is, of course, not a function, and hence we can't even consider it to be our sought-after inverse of the exponential function. Let's try to make things well defined.

Definition 3.7. Any function $\mathcal{L}\text{og} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ which satisfies $\exp(\mathcal{L}\text{og}\, z) = z$ is *a branch of the logarithm*. Let Arg z denote that argument of z which is in $(-\pi, \pi]$ (the *principal argument of z*). Then the *principal logarithm* is defined as

$$\text{Log}\, z = \ln|z| + i\,\text{Arg}\, z \,.$$

The paragraph preceding this definition ensures that the principal logarithm is indeed a branch of the logarithm. Even better, the evaluation of any branch of the logarithm at z can only differ from $\text{Log } z$ by a multiple of $2\pi i$; the reason for this is once more the periodicity of the exponential function.

So what about the other equation $\mathcal{L}\text{og}(\exp z) = z$? Let's try the principal logarithm: Suppose $z = x + iy$, then

$$\text{Log}(\exp z) = \text{Log}\left(e^x e^{iy}\right) = \ln\left|e^x e^{iy}\right| + i\,\text{Arg}\left(e^x e^{iy}\right) = \ln e^x + i\,\text{Arg}\left(e^{iy}\right) = x + i\,\text{Arg}\left(e^{iy}\right).$$

The right-hand side is equal to $z = x + iy$ only if $y \in (-\pi, \pi]$. The same happens with any other branch of the logarithm $\mathcal{L}\text{og}$—there will always be some (in fact, many) y-values for which $\mathcal{L}\text{og}(\exp z) \neq z$.

To end our discussion of the logarithm on a happy note, we prove that *any* branch of the logarithm has the *same* derivative; one just has to be cautious about where each logarithm is analytic.

Theorem 3.10. *Suppose $\mathcal{L}\text{og}$ is a branch of the logarithm. Then $\mathcal{L}\text{og}$ is differentiable wherever it is continuous and*

$$\mathcal{L}\text{og}' z = \frac{1}{z}.$$

Proof. The idea is to apply Lemma 2.5 to exp and $\mathcal{L}\text{og}$, but we need to be careful about the domains of these functions, so that we get actual inverse functions. Suppose $\mathcal{L}\text{og}$ maps $\mathbb{C} \setminus \{0\}$ to G (this is typically a half-open strip; you might want to think about what it looks like if $\mathcal{L}\text{og} = \text{Log}$). We apply Lemma 2.5 with $f : G \to \mathbb{C} \setminus \{0\}$, $f(z) = \exp(z)$ and $g : \mathbb{C} \setminus \{0\} \to G$, $g(z) = \mathcal{L}\text{og}$: if $\mathcal{L}\text{og}$ is continuous at z then

$$\mathcal{L}\text{og}' z = \frac{1}{\exp'(\mathcal{L}\text{og } z)} = \frac{1}{\exp(\mathcal{L}\text{og } z)} = \frac{1}{z}. \qquad \square$$

We finish this section by defining complex exponentials. For two complex numbers a and b, the natural definition $a^b = \exp(b \log a)$ (which is a concept borrowed from calculus) would in general yield more than one value (Exercise 31), so it is not always useful. We turn instead to the principal logarithm and define the *principal value of a^b* as

$$a^b = \exp(b \,\text{Log } a).$$

A note about e. In calculus one proves the equivalence of the real exponential function (as given, for example, through a power series) and the function $f(x) = e^x$ where e is *Euler's*[3] *number* and can be defined, for example, as $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. With our definition of a^b, we can now make a similar remark about the complex exponential function. Because e is a positive real number and hence $\text{Arg } e = 0$, we obtain

$$e^z = \exp(z \,\text{Log } e) = \exp\left(z\left(\ln|e| + i\,\text{Arg } e\right)\right) = \exp\left(z \ln e\right) = \exp\left(z\right).$$

A word of caution: this only works out this nicely because we carefully defined a^b for complex numbers. Different definitions might lead to different outcomes of e^z versus $\exp z$!

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[3]Named after Leonard Euler (1707–1783). For more information about Euler, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Euler.html`.

Exercises

1. Show that if $f(z) = \frac{az+b}{cz+d}$ is a Möbius transformation then $f^{-1}(z) = \frac{dz-b}{-cz+a}$.

2. Show that the derivative of a Möbius transformation is never zero.

3. Prove that any Möbius transformation different from the identity map can have at most two fixed points. (A *fixed point* of a function f is a number z such that $f(z) = z$.)

4. Prove Proposition 3.2.

5. Show that the Möbius transformation $f(z) = \frac{1+z}{1-z}$ maps the unit circle (minus the point $z = 1$) onto the imaginary axis.

6. Suppose that f is analytic on the region G and $f(G)$ is a subset of the unit circle. Show that f is constant. (*Hint*: Consider the function $\frac{1+f(z)}{1-f(z)}$ and use Exercise 5 and a variation of Exercise 13 in Chapter 2.)

7. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix of complex numbers whose determinant $ad - bc$ is non-zero. Then we can define a corresponding Möbius transformation T_A by $T_A(z) = \frac{az+b}{cz+d}$. Show that $T_A \circ T_B = T_{A \cdot B}$. (Here $\circ$ denotes composition and $\cdot$ denotes matrix multiplication.)

8. Let $f(z) = \frac{2z}{z+2}$. Draw two graphs, one showing the following six sets in the z plane and the other showing their images in the w plane. Label the sets. (You should only need to calculate the images of $0, \pm 2, \infty$ and $-1 - i$; remember that Möbius transformations preserve angles.)

   (a) The x-axis, plus ∞.
   (b) The y-axis, plus ∞.
   (c) The line $x = y$, plus ∞.
   (d) The circle with radius 2 centered at 0.
   (e) The circle with radius 1 centered at 1.
   (f) The circle with radius 1 centered at -1.

9. Find Möbius transformations satisfying each of the following. Write your answers in standard form, as $\frac{az+b}{cz+d}$.

   (a) $1 \to 0, \ 2 \to 1, \ 3 \to \infty$. (Use the cross-ratio.)
   (b) $1 \to 0, \ 1+i \to 1, \ 2 \to \infty$. (Use the cross-ratio.)
   (c) $0 \to i, \ 1 \to 1, \ \infty \to -i$.

10. Let C be the circle with center $1+i$ and radius 1. Using the cross-ratio, with different choices of z_k, find two different Möbius transformations that transform C onto the real axis plus infinity. In each case, find the image of the center of the circle.

11. Let C be the circle with center 0 and radius 1. Find a Möbius transformation which transforms C onto C and transforms 0 to $\frac{1}{2}$.

12. Describe the image of the region under the transformation:

    (a) The disk $|z| < 1$ under $w = \frac{iz-i}{z+1}$.

    (b) The quadrant $x > 0$, $y > 0$ under $w = \frac{z-i}{z+i}$.

    (c) The strip $0 < x < 1$ under $w = \frac{z}{z-1}$.

13. The *Jacobian* of a transformation $u = u(x, y)$, $v = v(x, y)$ is the determinant of the matrix $\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$. Show that if $f = u + iv$ is analytic then the Jacobian equals $|f'(z)|^2$.

14. Find the fixed points in $\hat{\mathbb{C}}$ of $f(z) = \frac{z^2-1}{2z+1}$.

15. Find the Möbius transformation f:

    (a) f maps $0 \to 1$, $1 \to \infty$, $\infty \to 0$.

    (b) f maps $1 \to 1$, $-1 \to i$, $-i \to -1$.

    (c) f maps x-axis to $y = x$, y-axis to $y = -x$, and the unit circle to itself.

16. Suppose z_1, z_2 and z_3 are distinct points in $\hat{\mathbb{C}}$. Show that z is on the circle passing through by z_1, z_2 and z_3 if and only if $[z, z_1, z_2, z_3]$ is real or infinite.

17. Describe the images of the following sets under the exponential function $\exp(z)$:

    (a) the line segment defined by $z = iy$, $0 \le y \le 2\pi$.

    (b) the line segment defined by $z = 1 + iy$, $0 \le y \le 2\pi$.

    (c) the rectangle $\{z = x + iy \in \mathbb{C} : 0 \le x \le 1, 0 \le y \le 2\pi\}$.

18. Prove Lemma 3.8.

19. Prove Lemma 3.9.

20. Let $z = x + iy$ and show that

    (a) $\sin z = \sin x \cosh y + i \cos x \sinh y$.

    (b) $\cos z = \cos x \cosh y - i \sin x \sinh y$.

21. Let $z = x + iy$ and show that

    (a) $|\sin z|^2 = \sin^2 x + \sinh^2 y = \cosh^2 y - \cos^2 x$.

    (b) $|\cos z|^2 = \cos^2 x + \sinh^2 y = \cosh^2 y - \sin^2 x$.

    (c) If $\cos x = 0$ then $|\cot z|^2 = \frac{\cosh^2 y - 1}{\cosh^2 y} \le 1$.

    (d) If $|y| \ge 1$ then $|\cot z|^2 \le \frac{\sinh^2 y + 1}{\sinh^2 y} = 1 + \frac{1}{\sinh^2 y} \le 1 + \frac{1}{\sinh^2 1} \le 2$.

22. Show that $\tan(iz) = i \tanh z$.

23. Find the principal values of

(a) $\log i$.

(b) $(-1)^i$.

(c) $\log(1+i)$.

24. Is $\arg(\bar{z}) = -\arg(z)$ true for the multiple-valued argument? What about $\mathrm{Arg}(\bar{z}) = -\mathrm{Arg}(z)$ for the principal branch?

25. Is there a difference between the set of all values of $\log(z^2)$ and the set of all values of $2\log z$? (Try some fixed numbers for z.)

26. For each of the following functions, determine all complex numbers for which the function is analytic. If you run into a logarithm, use the principal value (unless stated otherwise).

(a) $\bar{z}^2$.

(b) $\frac{\sin z}{z^3+1}$.

(c) $\mathcal{L}og(z - 2i + 1)$ where $\mathcal{L}og(z) = \ln|z| + i\,\mathcal{A}rg(z)$ with $0 \le \mathcal{A}rg(z) < 2\pi$.

(d) $\exp(\bar{z})$.

(e) $(z - 3)^i$.

(f) i^{z-3}.

27. Find all solutions to the following equations:

(a) $\mathrm{Log}(z) = \frac{\pi}{2}i$.

(b) $\mathrm{Log}(z) = \frac{3\pi}{2}i$.

(c) $\exp(z) = \pi i$.

(d) $\sin z = \cosh 4$.

(e) $\cos z = 0$.

(f) $\sinh z = 0$.

(g) $\overline{\exp(iz)} = \exp(i\bar{z})$.

(h) $z^{1/2} = 1 + i$.

28. Find the image of the annulus $1 < |z| < e$ under the principal value of the logarithm.

29. Show that $|a^z| = a^{\mathrm{Re}\,z}$ if a is a positive real constant.

30. Fix $c \in \mathbb{C} \setminus \{0\}$. Find the derivative of $f(z) = z^c$.

31. Prove that $\exp(b\log a)$ is single-valued if and only if b is an integer. (Note that this means that complex exponentials don't clash with monomials z^n.) What can you say if b is rational?

32. Describe the image under $\exp$ of the line with equation $y = x$. To do this you should find an equation (at least parametrically) for the image (you can start with the parametric form $x = t$, $y = t$), plot it reasonably carefully, and explain what happens in the limits as $t \to \infty$ and $t \to -\infty$.

33. For this problem, $f(z) = z^2$.

(a) Show that the image of a circle centered at the origin is a circle centered at the origin.

(b) Show that the image of a ray starting at the origin is a ray starting at the origin.

(c) Let T be the figure formed by the horizontal segment from 0 to 2, the circular arc from 2 to $2i$, and then the vertical segment from $2i$ to 0. Draw T and $f(T)$.

(d) Is the right angle at the origin in part (c) preserved? Is something wrong here?

(*Hint*: Use polar coordinates.)

34. As in the previous problem, let $f(z) = z^2$. Let Q be the square with vertices at 0, 2, $2 + 2i$ and $2i$. Draw $f(Q)$ and identify the types of image curves corresponding to the segments from 2 to $2 + 2i$ and from $2 + 2i$ to $2i$. They are not parts of either straight lines or circles. (*Hint*: You can write the vertical segment parametrically as $z(t) = 2 + it$. Eliminate the parameter in $u + iv = f(z(t))$ to get a (u, v) equation for the image curve.)

Chapter 4

Integration

Everybody knows that mathematics is about miracles, only mathematicians have a name for them: theorems.
Roger Howe

4.1 Definition and Basic Properties

At first sight, complex integration is not really anything different from real integration. For a continuous complex-valued function $\phi : [a,b] \subset \mathbb{R} \to \mathbb{C}$, we define

$$\int_a^b \phi(t)\, dt = \int_a^b \operatorname{Re} \phi(t)\, dt + i \int_a^b \operatorname{Im} \phi(t)\, dt\,. \tag{4.1}$$

For a function which takes complex numbers as arguments, we integrate over a curve γ (instead of a real interval). Suppose this curve is parametrized by $\gamma(t)$, $a \leq t \leq b$. If one meditates about the substitution rule for real integrals, the following definition, which is based on (4.1) should come as no surprise.

Definition 4.1. Suppose γ is a smooth curve parametrized by $\gamma(t)$, $a \leq t \leq b$, and f is a complex function which is continuous on γ. Then we define the *integral of f on γ* as

$$\int_\gamma f = \int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt\,.$$

This definition can be naturally extended to *piecewise smooth* curves, that is, those curves γ whose parametrization $\gamma(t)$, $a \leq t \leq b$, is only piecewise differentiable, say $\gamma(t)$ is differentiable on the intervals $[a,c_1],[c_1,c_2],\ldots,[c_{n-1},c_n],[c_n,b]$. In this case we simply define

$$\int_\gamma f = \int_a^{c_1} f(\gamma(t))\gamma'(t)\, dt + \int_{c_1}^{c_2} f(\gamma(t))\gamma'(t)\, dt + \cdots + \int_{c_n}^b f(\gamma(t))\gamma'(t)\, dt\,.$$

In what follows, we'll usually state our results for smooth curves, bearing in mind that practically all can be extended to piecewise smooth curves.

Example 4.1. As our first example of the application of this definition we will compute the integral of the function $f(z) = \bar{z}^2 = \left(x^2 - y^2\right) - i(2xy)$ over several curves from the point $z = 0$ to the point $z = 1 + i$.

(a) Let γ be the line segment from $z = 0$ to $z = 1 + i$. A parametrization of this curve is $\gamma(t) = t + it$, $0 \leq t \leq 1$. We have $\gamma'(t) = 1 + i$ and $f(\gamma(t)) = (t - it)^2$, and hence

$$\int_\gamma f = \int_0^1 (t - it)^2 \, (1 + i) \, dt = (1 + i) \int_0^1 t^2 - 2it^2 - t^2 \, dt = -2i(1 + i)/3 = \frac{2}{3}(1 - i) \, .$$

(b) Let γ be the arc of the parabola $y = x^2$ from $z = 0$ to $z = 1 + i$. A parametrization of this curve is $\gamma(t) = t + it^2$, $0 \leq t \leq 1$. Now we have $\gamma'(t) = 1 + 2it$ and

$$f(\gamma(t)) = \left(t^2 - \left(t^2\right)^2\right) - i\, 2t \cdot t^2 = t^2 - t^4 - 2it^3 \, ,$$

whence

$$\int_\gamma f = \int_0^1 \left(t^2 - t^4 - 2it^3\right)(1 + 2it) \, dt = \int_0^1 t^2 + 3t^4 - 2it^5 \, dt = \frac{1}{3} + 3\frac{1}{5} - 2i\frac{1}{6} = \frac{14}{15} - \frac{i}{3} \, .$$

(c) Let γ be the union of the two line segments γ_1 from $z = 0$ to $z = 1$ and γ_2 from $z = 1$ to $z = 1 + i$. Parameterizations are $\gamma_1(t) = t$, $0 \leq t \leq 1$ and $\gamma_2(t) = 1 + it$, $0 \leq t \leq 1$. Hence

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f = \int_0^1 t^2 \cdot 1 \, dt + \int_0^1 (1 - it)^2 i \, dt = \frac{1}{3} + i \int_0^1 1 - 2it - t^2 \, dt$$

$$= \frac{1}{3} + i\left(1 - 2i\frac{1}{2} - \frac{1}{3}\right) = \frac{4}{3} + \frac{2}{3}i \, .$$

The complex integral has some standard properties, most of which follow from their real siblings in a straightforward way.

Proposition 4.1. *Suppose γ is a smooth curve, f and g are complex functions which are continuous on γ, and $c \in \mathbb{C}$.*

(a) $\int_\gamma (f + cg) = \int_\gamma f + c \int_\gamma g \, .$

(b) *If γ is parametrized by $\gamma(t)$, $a \leq t \leq b$, define the curve $-\gamma$ through $-\gamma(t) = \gamma(a + b - t)$, $a \leq t \leq b$. Then $\int_{-\gamma} f = -\int_\gamma f \, .$*

(c) *If γ_1 and γ_2 are curves so that γ_2 starts where γ_1 ends then define the curve $\gamma_1\gamma_2$ by following γ_1 to its end, and then continuing on γ_2 to its end. Then $\int_{\gamma_1\gamma_2} f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{\gamma_2} f(z) \, dz \, .$*

(d) $\left|\int_\gamma f\right| \leq \max_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma) \, .$

The curve $-\gamma$ defined in (b) is the curve that we obtain by traveling through γ in the opposite direction.

In (d) the *length* of a smooth curve γ with parametrization $\gamma(t)$, $a \leq t \leq b$, is defined as

$$\text{length}(\gamma) = \int_a^b \left|\gamma'(t)\right| \, dt \, .$$

We invite the reader to use some familiar curves to see that this definition gives what one would expect to be the length of a curve.

Proof. (a) follows directly from the definition of the integral and the properties of real integrals.

(b) follows with an easy real change of variables $s = a + b - t$:

$$\int_{-\gamma} f = \int_a^b f\left(\gamma(a+b-t)\right)\left(\gamma(a+b-t)\right)'\,dt = -\int_a^b f\left(\gamma(a+b-t)\right)\gamma'(a+b-t)\,dt$$

$$= \int_b^a f\left(\gamma(s)\right)\gamma'(s)\,ds = -\int_a^b f\left(\gamma(s)\right)\gamma'(s)\,ds = -\int_\gamma f\,.$$

For (c) we need a suitable parameterization $\gamma(t)$ for $\gamma_1\gamma_2$. If γ_1 has domain $[a_1, b_1]$ and γ_2 has domain $[a_2, b_2]$ then we can use

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{for } a_1 \le t \le b_1, \\ \gamma_2(t - b_1 + a_2) & \text{for } b_1 \le t \le b_1 + b_2 - a_2. \end{cases}$$

The fact that $\gamma_1(b_1) = \gamma_2(a_2)$ is necessary to make sure that this parameterization is piecewise smooth. Now we break the integral over $\gamma_1\gamma_2$ into two pieces and apply the simple change of variables $s = t - b_1 + a_2$:

$$\int_{\gamma_1\gamma_2} f(z)\,dz = \int_{a_1}^{b_1+b_2-a_2} f(\gamma(t))\gamma'(t)\,dt = \int_{a_1}^{b_1} f(\gamma(t))\gamma'(t)\,dt + \int_{b_1}^{b_1+b_2-a_2} f(\gamma(t))\gamma'(t)\,dt$$

$$= \int_{a_1}^{b_1} f(\gamma_1(t))\gamma'(t)\,dt + \int_{b_1}^{b_1+b_2-a_2} f(\gamma_2(t-b_1+a_2))\gamma_2'(t-b_1+a_2)\,dt$$

$$= \int_{a_1}^{b_1} f(\gamma_1(t))\gamma'(t)\,dt + \int_{a_2}^{b_2} f(\gamma_2(s))\gamma_2'(s)\,ds$$

$$= \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

Finally, to prove (d), let $\phi = \text{Arg} \int_\gamma f$. Then

$$\left|\int_\gamma f(z)\,dz\right| = \int_\gamma f(z)\,dz\,e^{-i\phi} = \text{Re}\left(\int_\gamma f(z)\,dz\,e^{-i\phi}\right) = \text{Re}\left(\int_a^b f(\gamma(t))\gamma'(t)e^{-i\phi}\,dt\right)$$

$$= \int_a^b \text{Re}\left(f(\gamma(t))e^{-i\phi}\gamma'(t)\right)\,dt \le \int_a^b \left|f(\gamma(t))e^{-i\phi}\gamma'(t)\right|\,dt = \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\le \max_{a\le t\le b}|f(\gamma(t))|\int_a^b |\gamma'(t)|\,dt = \max_{z\in\gamma}|f(z)|\cdot\text{length}(\gamma)\,. \qquad \square$$

4.2 Antiderivatives

Just like in the real case, one easy way to compute integrals is through knowing the *antiderivative* (or *primitive*) of the integrand f, that is, a function F such that $F' = f$. To be more precise, we say that f *has an antiderivative on* G if there exists a function F that is analytic on G, such that $F'(z) = f(z)$ for all $z \in G$.

Theorem 4.2. *Suppose $G \subseteq \mathbb{C}$ is open, γ is a smooth curve in G parametrized by $\gamma(t)$, $a \leq t \leq b$, f is continuous on G, and F is a primitive of f on G. Then*

$$\int_\gamma f = F\left(\gamma(b)\right) - F\left(\gamma(a)\right).$$

In particular, $\int_\gamma f$ is independent of the path $\gamma \subset G$ between $\gamma(a)$ and $\gamma(b)$.

Example 4.1 shows that a path-independent integral is quite special; it also says that the function $\bar{z}^2$ does not have an antiderivative in, for example, the region $\{z \in \mathbb{C} : |z| < 2\}$. (Actually, the function $\bar{z}^2$ does not have an antiderivative in any nonempty region—prove it!)

In the special case that γ is closed (that is, $\gamma(a) = \gamma(b)$), we immediately get the following nice consequence.

Corollary 4.3. *Suppose $G \subseteq \mathbb{C}$ is open, γ is a smooth closed curve in G, and f is continuous on G and has an antiderivative on G. Then*

$$\int_\gamma f = 0.$$

Proof of Theorem 4.2. An application of the chain rule shows

$$\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t),$$

and then we calculate

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)\, dt = \int_a^b F'(\gamma(t))\gamma'(t)\, dt = \int_a^b \frac{d}{dt}F(\gamma(t))\, dt = F\left(\gamma(b)\right) - F\left(\gamma(a)\right),$$

by Theorem 1.9 (the Fundamental Theorem of Calculus). $\qquad\square$

4.3 Cauchy's Theorem

We now turn to the central theorem of complex analysis. It is based on the following concept.

Definition 4.2. Suppose γ_1 and γ_2 are closed curves in the open set $G \subseteq \mathbb{C}$, parametrized by $\gamma_1(t)$, $0 \leq t \leq 1$ and $\gamma_2(t)$, $0 \leq t \leq 1$, respectively. Then γ_1 is *G-homotopic* to γ_2, in symbols $\gamma_1 \sim_G \gamma_2$, if there is a continuous function $h : [0,1]^2 \to G$ such that

$$\begin{aligned} h(t,0) &= \gamma_1(t), \\ h(t,1) &= \gamma_2(t), \\ h(0,s) &= h(1,s). \end{aligned}$$

The function $h(t,s)$ is called a *homotopy* and represents a curve for each fixed s, which is continuously transformed from γ_1 to γ_2. The last condition simply says that each of the curves $h(t,s)$, $0 \leq t \leq 1$ is closed. An example is depicted in Figure 4.1.

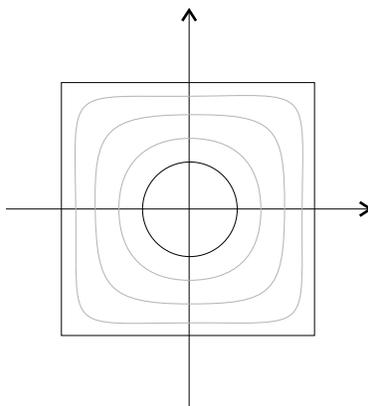Here is the theorem on which most of what will follow is based.

Figure 4.1: This square and the circle are $(\mathbb{C} \setminus \{0\})$-homotopic.

Theorem 4.4 (Cauchy's Theorem). *Suppose $G \subseteq \mathbb{C}$ is open, f is analytic in G, and $\gamma_1 \sim_G \gamma_2$ via a homotopy with continuous second partials. Then*

$$\int_{\gamma_1} f = \int_{\gamma_2} f \,.$$

Remarks. 1. The condition on the smoothness of the homotopy can be omitted, however, then the proof becomes too advanced for the scope of these notes. In all the examples and exercises that we'll have to deal with here, the homotopies will be 'nice enough' to satisfy the condition of this theorem.

2. It is assumed that Johann Carl Friedrich Gauß (1777–1855)[1] knew a version of this theorem in 1811 but only published it in 1831. Cauchy published his version in 1825, Weierstraß[2] his in 1842. Cauchy's theorem is often called the Cauchy–Goursat Theorem, since Cauchy assumed that the derivative of f was continuous, a condition which was first removed by Goursat[3].

An important special case is the one where a curve γ is G-homotopic to a point, that is, a constant curve (see Figure 4.2 for an example). In this case we simply say γ is *G-contractible*, in symbols $\gamma \sim_G 0$.

The fact that an integral over a point is zero has the following immediate consequence.

Corollary 4.5. *Suppose $G \subseteq \mathbb{C}$ is open, f is analytic in G, and $\gamma \sim_G 0$ via a homotopy with continuous second partials. Then*

$$\int_{\gamma} f = 0 \,.$$

---

[1]For more information about Gauß, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Gauss.html`.
[2]For more information about Karl Theodor Wilhelm Weierstraß (1815–1897), see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Weierstrass.html`.
[3]For more information about Edouard Jean-Baptiste Goursat (1858–1936), see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Goursat.html`.
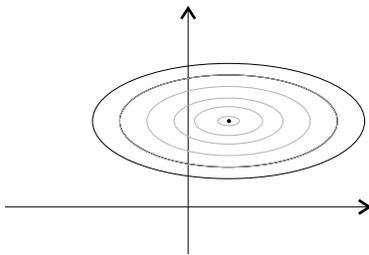
Figure 4.2: This ellipse is $(\mathbb{C} \setminus \mathbb{R})$-contractible.

The fact that any closed curve is $\mathbb{C}$-contractible (Exercise 11a) yields the following special case of the previous special-case corollary.

Corollary 4.6. *If f is entire and γ is any smooth closed curve then*

$$\int_\gamma f = 0\,.$$

Proof of Theorem 4.4. Suppose h is the homotopy, and γ_s is the curve parametrized by $h(t,s)$, $0 \leq t \leq 1$. Consider the integral

$$I(s) = \int_{\gamma_s} f$$

as a function in s (so $I(0) = \int_{\gamma_1} f$ and $I(1) = \int_{\gamma_2} f$). We will show that I is constant with respect to s, and hence the statement of the theorem follows with $I(0) = I(1)$. To prove that I is constant, we use Theorem 1.12 (Leibniz's rule), combined with Theorem 1.9 (the fundamental theorem of calculus).

$$
\begin{aligned}
\frac{d}{ds}I(s) &= \frac{d}{ds}\int_0^1 f\left(h(t,s)\right)\frac{\partial h}{\partial t}\,dt = \int_0^1 \frac{\partial}{\partial s}\left(f\left(h(t,s)\right)\frac{\partial h}{\partial t}\right)dt \\
&= \int_0^1 f'\left(h(t,s)\right)\frac{\partial h}{\partial s}\frac{\partial h}{\partial t} + f\left(h(t,s)\right)\frac{\partial^2 h}{\partial t \partial s}\,dt = \int_0^1 \frac{\partial}{\partial t}\left(f\left(h(t,s)\right)\frac{\partial h}{\partial s}\right)dt \\
&= f\left(h(1,s)\right)\frac{\partial h}{\partial s}(1,s) - f\left(h(0,s)\right)\frac{\partial h}{\partial s}(0,s) = 0\,.
\end{aligned}
$$

In the last step we used the third property (according to Definition 4.2) of the homotopy h. Note also that in the second line, we use the fact that h has continuous second partials and hence

$$\frac{\partial^2 h}{\partial t \partial s} = \frac{\partial^2 h}{\partial s \partial t}\,. \qquad \square$$

4.4 Cauchy's Integral Formula

Cauchy's Theorem 4.4 yields almost immediately the following helpful result.

Theorem 4.7 (Cauchy's Integral Formula for a Circle)**.** *Let C_R be the counterclockwise circle with radius R centered at w and suppose f is analytic at each point of the closed disk D bounded by C_R. Then*

$$f(w) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z-w}\, dz\,.$$

Proof. All circles C_r with center w and radius r are homotopic in $D \setminus \{w\}$, and the function $f(z)/(z-w)$ is analytic in an open set containing $D \setminus \{w\}$. So Cauchy's Theorem 4.4, gives

$$\int_{C_R} \frac{f(z)}{z-w}\, dz = \int_{C_r} \frac{f(z)}{z-w}\, dz$$

Now by Exercise 8,

$$\int_{C_r} \frac{1}{z-w}\, dz = 2\pi i\,,$$

and we obtain with Proposition 4.1(d)

$$\left| \int_{C_R} \frac{f(z)}{z-w}\, dz - 2\pi i f(w) \right| = \left| \int_{C_r} \frac{f(z)}{z-w}\, dz - f(w) \int_{C_r} \frac{1}{z-w}\, dz \right| = \left| \int_{C_r} \frac{f(z)-f(w)}{z-w}\, dz \right|$$

$$\leq \max_{z \in C_r} \left| \frac{f(z)-f(w)}{z-w} \right| \operatorname{length}(C_r) = \max_{z \in C_r} \frac{|f(z)-f(w)|}{r}\, 2\pi r$$

$$= 2\pi \max_{z \in C_r} |f(z) - f(w)|\,.$$

On the right-hand side, we can now take r as small as we want, and—because f is continuous at w—this means we can make $|f(z) - f(w)|$ as small as we like. Hence the left-hand side has no choice but to be zero, which is what we claimed. $\qquad\square$

This is a useful theorem by itself, but it can be made more generally useful. For example, it will be important to have Cauchy's integral formula when w is anywhere *inside C_R*, not just at the *center* of C_R. In fact, in many cases in which a point w is inside a simple closed curve γ we can see a homotopy from γ to a small circle around w so that the homotopy misses w and remains in the region where f is analytic. In that case the theorem remains true, since, by Cauchy's theorem, the integral of $f(z)/(z-w)$ around γ is the same as the integral of $f(z)/(z-w)$ around a small circle centered at w, and Theorem 4.7 then applies to evaluate the integral. In this discussion we need to be sure that the orientation of the curve γ and the circle match. In general, we say a simple closed curve γ is *positively oriented* if it is parameterized so that the inside is on the *left* of γ. For a circle this corresponds to a counterclockwise orientation.

Here's the general form:

Theorem 4.8 (Cauchy's Integral Formula)**.** *Suppose f is analytic on the region G, $w \in G$, and γ is a positively oriented, simple, closed, smooth, G-contractible curve such that w is inside γ. Then*

$$f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-w}\, dz\,.$$

We have already indicated how to prove this, by combining Cauchy's theorem and the special case, Theorem 4.7. All we need is to find a homotopy in $G \setminus \{w\}$ between γ and a small circle with center at w. In all practical cases we can see immediately how to construct such a homotopy, but it is not at all clear how to do so in complete generality; in fact, it is not even clear how to make sense of the "inside" of γ in general. The justification for this is one of the first substantial theorems ever proved in topology. We can state it as follows:

Theorem 4.9 (Jordan Curve Theorem)**.** *If γ is a positively oriented, simple, closed curve in $\mathbb{C}$ then $\mathbb{C} \setminus \gamma$ consists of two connected open sets, the inside and the outside of γ. If a closed disk D centered at w lies inside γ then there is a homotopy γ_s from γ to the positively oriented boundary of D, and, for $0 < s < 1$, γ_s is inside γ and outside of D.*

This theorem, although "intuitively obvious," is surprisingly difficult to prove. The usual statement of the Jordan curve theorem does not contain the homotopy information; we have borrowed this from a companion theorem to the Jordan curve theorem which is sometimes called the "annulus theorem." If you want to explore this kind of mathematics you should take a course in topology.

A nice special case of Cauchy's formula is obtained when γ is a circle centered at w, parametrized by, say, $z = w + re^{it}$, $0 \le t \le 2\pi$. Theorem 4.8 gives (if the conditions are met)

$$f(w) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(w + re^{it}\right)}{w + re^{it} - w} ire^{it} \, dt = \frac{1}{2\pi} \int_0^{2\pi} f\left(w + re^{it}\right) dt \,.$$

Even better, we automatically get similar formulas for the real and imaginary part of f, simply by taking real and imaginary parts on both sides. These identities have the flavor of mean values. Let's summarize them in the following statement, which is often called a *mean-value theorem.*

Corollary 4.10. *Suppose f is analytic on and inside the circle $z = w + re^{it}$, $0 \le t \le 2\pi$. Then*

$$f(w) = \frac{1}{2\pi} \int_0^{2\pi} f\left(w + re^{it}\right) dt \,.$$

Furthermore, if $f = u + iv$,

$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} u\left(w + re^{it}\right) dt \qquad and \qquad v(w) = \frac{1}{2\pi} \int_0^{2\pi} v\left(w + re^{it}\right) dt \,.$$

Exercises

1. Integrate the function $f(z) = \overline{z}$ over the three curves given in Example 4.1.

2. Evaluate $\int_\gamma \frac{1}{z} \, dz$ where $\gamma(t) = \sin t + i \cos t$, $0 \le t \le 2\pi$.

3. Integrate the following functions over the circle $|z| = 2$, oriented counterclockwise:

    (a) $z + \overline{z}$.
    (b) $z^2 - 2z + 3$.
    (c) $1/z^4$.
    (d) xy.

4. Evaluate the integrals $\int_\gamma x\,dz$, $\int_\gamma y\,dz$, $\int_\gamma z\,dz$ and $\int_\gamma \bar{z}\,dz$ along each of the following paths. Note that you can get the second two integrals very easily after you calculate the first two, by writing z and $\bar{z}$ as $x \pm iy$.

   (a) γ is the line segment form 0 to $1 - i$.

   (b) γ is the counterclockwise circle $|z| = 1$.

   (c) γ is the counterclockwise circle $|z - a| = r$. Use $\gamma(t) = a + re^{it}$.

5. Evaluate $\int_\gamma e^{3z}\,dz$ for each of the following paths:

   (a) The straight line segment from 1 to i.

   (b) The circle $|z| = 3$.

   (c) The parabola $y = x^2$ from $x = 0$ to $x = 1$.

6. Evaluate $\int_\gamma \left|z^2\right|\,dz$ where γ is the parabola with parametric equation $\gamma(t) = t + it^2$, $0 \le t \le 1$.

7. Evaluate $\int_\gamma z^{\frac{1}{2}}\,dz$ where γ is the unit circle and $z^{\frac{1}{2}}$ is the principal branch. You can use the parameterization $\gamma(\theta) = e^{i\theta}$ for $-\pi \le \theta \le \pi$, and remember that the principal branch is defined by $z^{\frac{1}{2}} = \sqrt{r}e^{i\theta/2}$ if $z = re^{i\theta}$ for $-\pi \le \theta \le \pi$.

8. Let γ be the circle with radius r centered at w, oriented counterclockwise. You can parameterize this curve as $z(t) = w + re^{it}$ for $0 \le t \le 2\pi$. Show that

$$\int_\gamma \frac{dz}{z - w} = 2\pi i\,.$$

9. Suppose a smooth curve is parametrized by both $\gamma(t)$, $a \le t \le b$ and $\sigma(t)$, $c \le t \le d$, and let $\tau : [c, d] \to [a, b]$ be the map which "takes γ to σ," that is, $\sigma = \gamma \circ \tau$. Show that

$$\int_c^d f(\sigma(t))\sigma'(t)\,dt = \int_a^b f(\gamma(t))\gamma'(t)\,dt\,.$$

(In other words, our definition of the integral $\int_\gamma f$ is independent of the parametrization of γ.)

10. Prove that $\sim_G$ is an equivalence relation.

11. (a) Prove that any closed curve is $\mathbb{C}$-contractible.

   (b) Prove that any two closed curves are $\mathbb{C}$-homotopic.

12. Show that $\int_\gamma z^n\,dz = 0$ for any closed smooth γ and any integer $n \ne -1$. [If n is negative, assume that γ does not pass through the origin, since otherwise the integral is not defined.]

13. Exercise 12 excluded $n = -1$ for a very good reason: Exercises 2 and 8 (with $w = 0$) give counterexamples. Generalizing these, if m is any integer then find a closed curve γ so that $\int_\gamma z^{-1}\,dz = 2m\pi i$. (*Hint*: Follow the counterclockwise unit circle through m complete cycles (for $m > 0$). What should you do if $m < 0$? What if $m = 0$?)

14. Let γ_r be the circle centered at $2i$ with radius r, oriented counterclockwise. Compute

$$\int_{\gamma_r} \frac{dz}{z^2 + 1}.$$

(This integral depends on r.)

15. Suppose p is a polynomial and γ is a closed smooth path in $\mathbb{C}$. Show that

$$\int_\gamma p = 0.$$

16. Compute the *real* integral

$$\int_0^{2\pi} \frac{d\theta}{2 + \sin\theta}$$

by writing the sine function in terms of the exponential function and making the substitution $z = e^{i\theta}$ to turn the real into a complex integral.

17. Show that $F(z) = \frac{i}{2}\text{Log}(z+i) - \frac{i}{2}\text{Log}(z-i)$ is a primitive of $\frac{1}{1+z^2}$ for $\text{Re}(z) > 0$. Is $F(z) = \arctan z$?

18. Prove the following integration by parts statement. Let f and g be analytic in G, and suppose $\gamma \subset G$ is a smooth curve from a to b. Then

$$\int_\gamma fg' = f(\gamma(b))g(\gamma(b)) - f(\gamma(a))g(\gamma(a)) - \int_\gamma f'g.$$

19. Suppose f and g are analytic on the region G, γ is a closed, smooth, G-contractible curve, and $f(z) = g(z)$ for all $z \in \gamma$. Prove that $f(z) = g(z)$ for all z inside γ.

20. This exercise gives an alternative proof of Cauchy's integral formula (Theorem 4.8), which does not depend on Cauchy's Theorem 4.4. Suppose f is analytic on the region G, $w \in G$, and γ is a positively oriented, simple, closed, smooth, G-contractible curve such that w is inside γ.

    (a) Consider the function $g : [0, 1] \to \mathbb{C}$, $g(t) = \int_\gamma \frac{f(w+t(z-w))}{z-w} dz$. Show that $g' = 0$. (*Hint*: Use Theorem 1.12 (Leibniz's rule) and then find a primitive for $\frac{\partial f}{\partial t}(z + t(w - z))$.)

    (b) Prove Theorem 4.8 by evaluating $g(0)$ and $g(1)$.

21. Prove Corollary 4.5 using Theorem 4.8.

22. Suppose a is a complex number and γ_0 and γ_1 are two counterclockwise circles (traversed just once) so that a is inside both of them. Explain geometrically why γ_0 and γ_1 are homotopic in $\mathbb{C} \setminus \{a\}$.

23. Let γ_r be the counterclockwise circle with center at 0 and radius r. Find $\int_{\gamma_r} \frac{dz}{z-a}$. You should get different answers for $r < |a|$ and $r > |a|$. (*Hint*: In one case γ_r is contractible in $\mathbb{C} \setminus \{a\}$. In the other you can combine Exercises 8 and 22.)

24. Let γ_r be the counterclockwise circle with center at 0 and radius r. Find $\int_{\gamma_r} \frac{dz}{z^2-2z-8}$ for $r = 1$, $r = 3$ and $r = 5$. (*Hint*: Since $z^2 - 2z - 8 = (z - 4)(z + 2)$ you can find a partial fraction decomposition of the form $\frac{1}{z^2-2z-8} = \frac{A}{z-4} + \frac{B}{z+2}$. Now use Exercise 23.)

25. Use the Cauchy integral formula to evaluate the integral in Exercise 24 when $r = 3$. (*Hint*: The integrand can be written in each of following ways:

$$\frac{1}{z^2 - 2z - 8} = \frac{1}{(z-4)(z+2)} = \frac{1/(z-4)}{z+2} = \frac{1/(z+2)}{z-4} .$$

Which of these forms corresponds to the Cauchy integral formula for the curve γ_3?)

26. Compute the following integrals, where C is the boundary of the square with corners at $\pm 4 \pm 4i$:

(a) $\displaystyle\int_C \frac{e^z}{z^3}\, dz.$

(b) $\displaystyle\int_C \frac{e^z}{(z - \pi i)^4}\, dz.$

(c) $\displaystyle\int_C \frac{\sin(2z)}{(z - \pi)^4}\, dz.$

(d) $\displaystyle\int_C \frac{e^z \cos(z)}{(z - \pi)^3}\, dz.$

Chapter 5

Consequences of Cauchy's Theorem

> *If things are nice there is probably a good reason why they are nice: and if you do not know at least one reason for this good fortune, then you still have work to do.*
> Richard Askey

5.1 Extensions of Cauchy's Formula

We now derive formulas for f' and f'' which resemble Cauchy's formula (Theorem 4.8).

Theorem 5.1. *Suppose f is analytic on the region G, $w \in G$, and γ is a positively oriented, simple, closed, smooth, G-contractible curve such that w is inside γ. Then*

$$f'(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^2} \, dz$$

and

$$f''(w) = \frac{1}{\pi i} \int_\gamma \frac{f(z)}{(z-w)^3} \, dz \,.$$

This innocent-looking theorem has a very powerful consequence: just from knowing that f is analytic we know of the existence of f'', that is, f' *is also analytic in G*. Repeating this argument for f', then for f'', f''', etc., gives the following statement, which has no analog whatsoever in the reals.

Corollary 5.2. *If f is differentiable in the region G then f is infinitely differentiable in G.*

Proof of Theorem 5.1. The idea of the proof is very similar to the proof of Cauchy's integral formula (Theorem 4.8). We will study the following difference quotient, which we can rewrite as follows by Theorem 4.8.

$$
\begin{aligned}
\frac{f(w + \Delta w) - f(w)}{\Delta w} &= \frac{1}{\Delta w} \left(\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - (w + \Delta w)} \, dz - \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} \, dz \right) \\
&= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - w - \Delta w)(z - w)} \, dz \,.
\end{aligned}
$$

Hence we will have to show that the following expression gets arbitrarily small as $\Delta w \to 0$:

$$\frac{f(w + \Delta w) - f(w)}{\Delta w} - \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^2}\, dz = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - w - \Delta w)(z - w)} - \frac{f(z)}{(z-w)^2}\, dz$$

$$= \Delta w \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - w - \Delta w)(z-w)^2}\, dz\,.$$

This can be made arbitrarily small if we can show that the integral stays bounded as $\Delta w \to 0$. In fact, by Proposition 4.1(d), it suffices to show that the *integrand* stays bounded as $\Delta w \to 0$ (because γ and hence length(γ) are fixed). Let $M = \max_{z \in \gamma} |f(z)|$. Since γ is a closed set, there is some positive δ so that the open disk of radius δ around w does not intersect γ; that is, $|z - w| \geq \delta$ for all z on γ. By the reverse triangle inequality we have for all $z \in \gamma$

$$\left| \frac{f(z)}{(z - w - \Delta w)(z - w)^2} \right| \leq \frac{|f(z)|}{(|z - w| - |\Delta w|)|z - w|^2} \leq \frac{M}{(\delta - |\Delta w|)\delta^2}\,,$$

which certainly stays bounded as $\Delta w \to 0$. The proof of the formula for f'' is very similar and will be left for the exercises (see Exercise 1). □

Remarks. 1. Theorem 5.1 suggests that there are similar looking formulas for the higher derivatives of f. This is in fact true, and theoretically one could obtain them one by one with the methods of the proof of Theorem 5.1. However, once we start studying power series for analytic functions, we will obtain such a result much more easily; so we save the derivation of formulas for higher derivatives of f for later (see Corollary 8.6).

2. Theorem 5.1 can also be used to compute certain integrals. We give some examples of this application next.

Example 5.1.
$$\int_{|z|=1} \frac{\sin(z)}{z^2}\, dz = 2\pi i\, \frac{d}{dz} \sin(z) \bigg|_{z=0} = 2\pi i \cos(0) = 2\pi i\,.$$

Example 5.2. To compute the integral

$$\int_{|z|=2} \frac{dz}{z^2(z-1)}\,,$$

we first split up the integration path as illustrated in Figure 5.1: Introduce an additional path which separates 0 and 1. If we integrate on these two new closed paths (γ_1 and γ_2) counterclockwise, the two contributions along the new path will cancel each other. The effect is that we transformed an integral, for which two singularities where inside the integration path, into a sum of two integrals, each of which has only one singularity inside the integration path; these new integrals we know
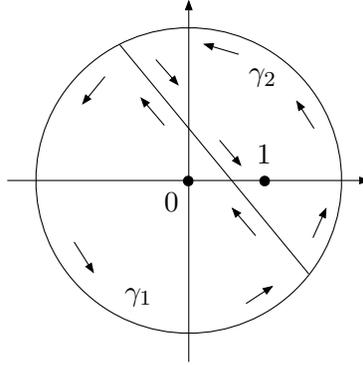
Figure 5.1: Example 5.2

how to deal with.

$$\int_{|z|=2} \frac{dz}{z^2(z-1)} = \int_{\gamma_1} \frac{dz}{z^2(z-1)} + \int_{\gamma_2} \frac{dz}{z^2(z-1)}$$

$$= \int_{\gamma_1} \frac{\frac{1}{z-1}}{z^2}\, dz + \int_{\gamma_2} \frac{\frac{1}{z^2}}{z-1}\, dz$$

$$= 2\pi i\, \frac{d}{dz}\frac{1}{z-1}\bigg|_{z=0} + 2\pi i \frac{1}{1^2}$$

$$= 2\pi i\left(-\frac{1}{(-1)^2}\right) + 2\pi i$$

$$= 0\,.$$

Example 5.3.

$$\int_{|z|=1} \frac{\cos(z)}{z^3}\, dz = \pi i\, \frac{d^2}{dz^2}\cos(z)\bigg|_{z=0} = \pi i\left(-\cos(0)\right) = -\pi i\,.$$

5.2 Taking Cauchy's Formula to the Limit

Many beautiful applications of Cauchy's formula arise from considerations of the limiting behavior of the formula as the curve gets arbitrarily large. We shall look at a few applications along these lines in this section, but this will be a recurring theme throughout the rest of the book.

The first application is understanding the roots of polynomials. As a preparation we prove the following inequality, which is generally quite useful. It simply says that for large enough z, a polynomial of degree d looks almost like a constant times z^d.

Lemma 5.3. *Suppose $p(z)$ is a polynomial of degree d with leading coefficient a_d. Then there is real number R_0 so that*

$$\frac{1}{2}\,|a_d|\,|z|^d \le |p(z)| \le 2\,|a_d|\,|z|^d$$

for all z satisfying $|z| \ge R_0$.

Proof. Since $p(z)$ has degree d its leading coefficient a_d is not zero, and we can factor out $a_d z^d$:

$$|p(z)| = \left| a_d z^d + a_{d-1} z^{d-1} + a_{d-2} z^{d-2} + \cdots + a_1 z + a_0 \right|$$

$$= |a_d| \, |z|^d \left| 1 + \frac{a_{d-1}}{a_d z} + \frac{a_{d-2}}{a_d z^2} + \cdots + \frac{a_1}{a_d z^{d-1}} + \frac{a_0}{a_d z^d} \right|.$$

Then the sum inside the last factor has limit 1 as $z \to \infty$ so its modulus is between $\frac{1}{2}$ and 2 for all large enough z. □

Theorem 5.4 (Fundamental theorem of algebra[1]). *Every non-constant polynomial has a root in $\mathbb{C}$.*

Proof[2]. Suppose (by way of contradiction) that p does not have any roots, that is, $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then Cauchy's formula gives us

$$\frac{1}{p(0)} = \frac{1}{2\pi i} \int_{\gamma_R} \frac{1/p(z)}{z} \, dz$$

where γ_R is the circle of radius R around the origin. Notice that the value of the integral does not depend on R, so we have

$$\frac{1}{p(0)} = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\gamma_R} \frac{dz}{z \, p(z)}. \tag{$*$}$$

But now we can see that the limit of the integral is 0: By Lemma 5.3 we have $|z \, p(z)| \geq \frac{1}{2} |a_d| \, |z|^{d+1}$ for all large z, where d is the degree of $p(z)$ and a_d is the leading coefficient of $p(z)$. Hence, using Proposition 4.1(d) and the formula for the circumference of a circle we see that the integral can be bounded as

$$\left| \frac{1}{2\pi i} \int_{\gamma_R} \frac{dz}{z p(z)} \right| \leq \frac{1}{2\pi} \cdot \frac{2}{|a_d| \, R^{d+1}} \cdot (2\pi R) = \frac{2}{|a_d| \, R^d}$$

and this has limit 0 as $R \to \infty$. But, plugging into $(*)$, we have shown that $\frac{1}{p(0)} = 0$, which is impossible. □

Remarks. 1. This statement implies that any polynomial p can be factored into linear terms of the form $z - a$ where a is a root of p, as we can apply the corollary, after getting a root a, to $\frac{p(z)}{z-a}$ (which is again a polynomial by the division algorithm), etc. (see also Exercise 8).

2. A compact reformulation of the fundamental theorem of algebra is to say that $\mathbb{C}$ is *algebraically closed.*

A powerful consequence of (the first half of) Theorem 5.1 is the following.

Corollary 5.5 (Liouville's[3] Theorem[4]). *Every bounded entire function is constant.*

---

[1]The fundamental theorem of algebra was first proved by Gauß (in his doctoral dissertation), although its statement had been assumed to be correct long before Gauß's times.

[3]For more information about Joseph Liouville (1809–1882), see
http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Liouville.html.

[4]This theorem is for historical reasons erroneously attributed to Liouville. It was published earlier by Cauchy; in fact, Gauß may well have known about it before Cauchy's times.

Proof. Suppose $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Given any $w \in \mathbb{C}$, we apply Theorem 5.1 with the circle γ_R of radius R centered at w. Note that we can choose any R because f is entire. Now we apply Proposition 4.1 (d), remembering that γ_R has circumference $2\pi R$ and $|z - w| = R$ for all z on γ_R:

$$|f'(w)| = \left| \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z-w)^2} \, dz \right| \leq \frac{1}{2\pi} \max_{z \in \gamma_R} \left| \frac{f(z)}{(z-w)^2} \right| \cdot 2\pi R = \frac{1}{2\pi} \max_{z \in \gamma_R} \frac{|f(z)|}{R^2} 2\pi R = \max_{z \in \gamma} \frac{|f(z)|}{R}$$

$$\leq \frac{M}{R}.$$

The right-hand side can be made arbitrary small, as we are allowed to make R as large as we want. This implies that $f' = 0$, and hence, by Theorem 2.7, f is constant. $\square$

As an example of the usefulness of Liouville's theorem we give another proof of the fundamental theorem of algebra, which is close to Gauß's original proof:

Another proof of the fundamental theorem of algebra. Suppose (by way of contradiction) that p does not have any roots, that is, $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then, because p is entire, the function $f(z) = \frac{1}{p(z)}$ is entire. But $f \to 0$ as $|z|$ becomes large as a consequence of Lemma 5.3; that is, f is also bounded (Exercise 7). Now apply Corollary 5.5 to deduce that f is constant. Hence p is constant, which contradicts our assumptions. $\square$

As one more example of this theme of getting results from Cauchy's formula by taking the limit as a path goes to infinity, we compute an improper integral.

Let σ be the counterclockwise semicircle formed by the segment S of the real axis from $-R$ to R, followed by the circular arc T of radius R in the upper half plane from R to $-R$, where $R > 1$. We shall integrate the function

$$f(z) = \frac{1}{z^2 + 1} = \frac{1/(z+i)}{z - i} = \frac{g(z)}{z - i}, \text{ where } g(z) = \frac{1}{z + i}$$

Since $g(z)$ is analytic inside and on σ and i is inside σ, we can apply Cauchy's formula:

$$\frac{1}{2\pi i} \int_\sigma \frac{dz}{z^2 + 1} = \frac{1}{2\pi i} \int_\sigma \frac{g(z)}{z - i} \, dz = g(i) = \frac{1}{i + i} = \frac{1}{2i},$$

and so

$$\int_S \frac{dz}{z^2 + 1} + \int_T \frac{dz}{z^2 + 1} = \int_\sigma \frac{dz}{z^2 + 1} = 2\pi i \cdot \frac{1}{2i} = \pi. \tag{$**$}$$

Now this formula holds for all $R > 1$, so we can take the limit as $R \to \infty$. First, $|z^2 + 1| \geq \frac{1}{2} |z|^2$ for large enough z by Lemma 5.3, so we can bound the integral over T using Proposition 4.1(d):

$$\left| \int_T \frac{dz}{z^2 + 1} \right| \leq \frac{2}{R^2} \cdot \pi R = \frac{2}{R}$$

and this has limit 0 as $R \to \infty$. On the other hand, we can parameterize the integral over S using $z = t$, $-R \leq t \leq R$, obtaining

$$\int_S \frac{dz}{z^2 + 1} = \int_{-R}^R \frac{dt}{1 + t^2}.$$

As $R \to \infty$ this approaches an improper integral. Making these observations in the limit of the formula $(**)$ as $R \to \infty$ now produces

$$\int_{-\infty}^{\infty} \frac{dt}{t^2 + 1} = \pi.$$

Of course this integral can be evaluated almost as easily using standard formulas from calculus. However, just a slight modification of this example leads to an improper integral which is far beyond the scope of basic calculus; see Exercise 11.

5.3 Antiderivatives Revisited and Morera's Theorem

A region G is said to be *simply connected* if every closed curve in G is G-contractible. This concept allows the following result.

Theorem 5.6. *Suppose f is analytic in the simply-connected region G. Then f has a primitive in G.*

Proof. Fix a point $a \in G$ and let

$$F(z) = \int_{\gamma_z} f$$

where γ_z is any smooth curve from a to z. We should make sure that F is well defined: Suppose δ_z is another smooth curve from a to z then $\gamma_z - \delta_z$ is closed and G-contractible, as G is simply connected. Hence by Corollary 4.5

$$0 = \int_{\gamma_z - \delta_z} f = \int_{\gamma_z} f - \int_{\delta_z} f$$

which means we get the same integral no matter which path we take from a to z, so F is a well-defined function. It remains to show that F is a primitive of f:

$$F'(z) = \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = \lim_{h \to 0} \frac{1}{h} \left(\int_{\gamma_{z+h}} f - \int_{\gamma_z} f \right).$$

Now let δ be a smooth curve in G from z to $z + h$. Then $\gamma_z + \delta - \gamma_{z+h}$ is a closed smooth curve in G, and it is G-contractible as G is simply connected. Hence again Corollary 4.5 gives us

$$\int_{\gamma_z} f + \int_{\delta} f - \int_{\gamma_{z+h}} f = 0,$$

that is,

$$F'(z) = \lim_{h \to 0} \frac{1}{h} \left(\int_{\gamma_{z+h}} f - \int_{\gamma_z} f \right) = \lim_{h \to 0} \frac{1}{h} \int_{\delta} f.$$

(One should keep in mind that δ very much depends on z and h.) If h is sufficiently small, the line segment $l(z, z+h)$ between z and $z + h$ lies in G and by Corollary 4.5 (again we use that G is simply connected)

$$F'(z) = \lim_{h \to 0} \frac{1}{h} \int_{\delta} f = \lim_{h \to 0} \frac{1}{h} \int_{l(z,z+h)} f.$$

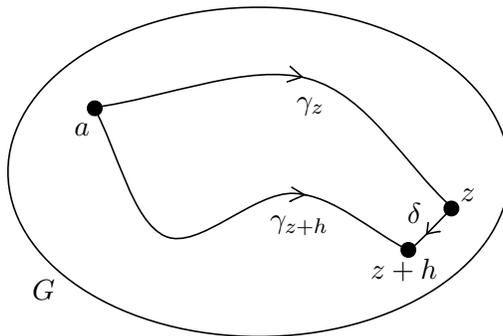Figure 5.2: Proof of Theorem 5.6.

Now because $\displaystyle\int_{l(z,z+h)} f(z)\,dw = f(z)\int_{l(z,z+h)} dw = f(z)\,h$, we obtain

$$
\begin{aligned}
\left|F'(z) - f(z)\right| &= \left|\lim_{h\to 0}\frac{1}{h}\int_{l(z,z+h)} f(w)\,dw - \lim_{h\to 0}\frac{1}{h}\int_{l(z,z+h)} f(z)\,dw\right| \\
&= \lim_{h\to 0}\left|\frac{1}{h}\int_{l(z,z+h)} f(w) - f(z)\,dw\right| \\
&\leq \lim_{h\to 0}\frac{1}{|h|}\max_{w\in l(z,z+h)}|f(w) - f(z)|\,\operatorname{length}(l(z,z+h)) \\
&= \lim_{h\to 0}\max_{w\in l(z,z+h)}|f(w) - f(z)| = 0\,.
\end{aligned}
$$

The last equality follows from the continuity of f. $\qquad\square$

There is an interesting consequence to be drawn from this theorem. It follows from the fact that a primitive of a function is, by definition, differentiable. This means that the primitive of a function f obtained by Theorem 5.6 has itself a primitive, which has a primitive, which has a primitive, which has ... This is the same behavior which we discovered in Corollary 5.2 'in the other direction.'

Another consequence comes from the *proof* of Theorem 5.6: we did not really need the fact that every closed curve in G is contractible, just that every closed curve gives a zero integral for f. This fact can be exploited to give a sort of converse statement to Corollary 4.5.

Corollary 5.7 (Morera's[5] Theorem)**.** *Suppose f is continuous in the region G and*

$$
\int_\gamma f = 0
$$

for all smooth closed paths $\gamma \subset G$. Then f is analytic in G.

---

[5]For more information about Giancinto Morera (1856–1907), see
http://www-groups.dcs.st-and.ac.uk/∼history/Biographies/Morera.html.

Proof. As in the previous proof, we fix an $a \in G$ and define

$$F(z) = \int_{\gamma_z} f \,,$$

where γ_z is any smooth curve in G from a to z. As above, this is a well-defined function because all closed paths give a zero integral for f; and exactly as above we can show that F is a primitive for f in G. Because F is analytic on G, Corollary 5.2 gives that f is also analytic on G. $\qquad \square$

Exercises

1. Prove the formula for f'' in Theorem 5.1.

2. Integrate the following functions over the circle $|z| = 3$, oriented counterclockwise:

   (a) $\mathrm{Log}(z - 4i)$.

   (b) $\frac{1}{z - \frac{1}{2}}$.

   (c) $\frac{1}{z^2 - 4}$.

   (d) $\frac{\exp z}{z^3}$.

   (e) $\left(\frac{\cos z}{z}\right)^2$.

   (f) i^{z-3}.

   (g) $\frac{\sin z}{(z^2 + \frac{1}{2})^2}$.

   (h) $\frac{\exp z}{(z-w)^2}$, where w is any fixed complex number with $|w| \neq 3$.

   (i) $\frac{1}{(z+4)(z^2+1)}$.

3. Prove that $\int_\gamma z \exp\left(z^2\right) dz = 0$ for any closed curve γ.

4. Show that $\exp(\sin z)$ has an antiderivative on $\mathbb{C}$.

5. Find a (maximal size) set on which $f(z) = \exp\left(\frac{1}{z}\right)$ has an antiderivative. (How does this compare with the real function $f(x) = e^{1/x}$?)

6. Compute the following integrals; use the principal value of z^i. (*Hint*: one of these integrals is considerably easier than the other.)

   (a) $\displaystyle\int_{\gamma_1} z^i \, dz$ where $\gamma_1(t) = e^{it}, \ -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

   (b) $\displaystyle\int_{\gamma_2} z^i \, dz$ where $\gamma_2(t) = e^{it}, \ \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$.

7. Suppose f is continuous on $\mathbb{C}$ and $\lim_{|z| \to \infty} f(z) = 0$. Show that f is bounded. (*Hint*: From the definition of limit at infinity (with $\epsilon = 1$) there is $R > 0$ so that $|f(z) - 0| = |f|(z) < 1$ if $|z| > R$. Is f bounded for $|z| \leq R$?)

8. Let p be a polynomial of degree $n > 0$. Prove that there exist complex numbers $c, z_1, z_2, \ldots, z_k$ and positive integers $j_1, \ldots, j_k$ such that

$$p(z) = c\,(z - z_1)^{j_1}\,(z - z_2)^{j_2}\cdots(z - z_k)^{j_k}\ ,$$

where $j_1 + \cdots + j_k = n$.

9. Show that a polynomial of odd degree with real coefficients must have a real zero. (*Hint*: Exercise 14b in Chapter 1.)

10. Suppose f is entire and there exist constants a, b such that $|f(z)| \le a|z| + b$ for all $z \in \mathbb{C}$. Prove that f is a linear polynomial (that is, of degree ≤ 1).

11. In this problem $F(z) = \frac{e^{iz}}{z^2+1}$ and $R > 1$. Modify the example at the end of Section 5.2:

    (a) Show that $\int_\sigma F(z)\,dz = \frac{\pi}{e}$ if σ is the counterclockwise semicircle formed by the segment S of the real axis from $-R$ to R, followed by the circular arc T of radius R in the upper half plane from R to $-R$.

    (b) Show that $\left|e^{iz}\right| \le 1$ for z in the upper half plane, and conclude that $|F(z)| \le \frac{2}{|z|^2}$ for z large enough.

    (c) Show that $\lim_{R\to\infty} \int_T F(z)\,dz = 0$, and hence $\lim_{R\to\infty} \int_S F(z)\,dz = \frac{\pi}{e}$.

    (d) Conclude, by parameterizing the integral over S in terms of t and just considering the real part, that $\int_{-\infty}^{\infty} \frac{\cos(t)}{t^2+1}\,dx = \frac{\pi}{e}$.

Chapter 6

Harmonic Functions

> *The shortest route between two truths in the real domain passes through the complex domain.*
> J. Hadamard

6.1 Definition and Basic Properties

We will now spend a chapter on certain functions defined on subsets of the complex plane which are *real* valued. The main motivation for studying them is that the partial differential equation they satisfy is very common in the physical sciences.

Definition 6.1. Let $G \subseteq \mathbb{C}$ be a region. A function $u : G \to \mathbb{R}$ is *harmonic* in G if it has continuous second partials in G and satisfies the *Laplace*[1] *equation*

$$u_{xx} + u_{yy} = 0$$

in G.

There are (at least) two reasons why harmonic functions are part of the study of complex analysis, and they can be found in the next two theorems.

Proposition 6.1. *Suppose $f = u + iv$ is analytic in the region G. Then u and v are harmonic in G.*

Proof. First, by Corollary 5.2, f is infinitely differentiable, and hence so are u and v. In particular, u and v have continuous second partials. By Theorem 2.6, u and v satisfy the Cauchy–Riemann equations

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x$$

in G. Hence

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

in G. Note that in the last step we used the fact that v has continuous second partials. The proof that v satisfies the Laplace equation is completely analogous. $\qquad\square$

---

[1]For more information about Pierre-Simon Laplace (1749–1827), see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Laplace.html`.

Proposition 6.1 shouts for a converse theorem. There are, however, functions which are harmonic in a region G but not the real part (say) of an analytic function in G (Exercise 3). We do obtain a converse of Proposition 6.1 if we restrict ourselves to *simply connected* regions.

Theorem 6.2. *Suppose u is harmonic on the simply connected region G. Then there exists a harmonic function v such that $f = u + iv$ is analytic in G.*

Remark. The function v is called a *harmonic conjugate* of u.

Proof. We will explicitly construct the analytic function f (and thus $v = \text{Im } f$). First, let

$$g = u_x - iu_y \,.$$

The plan is to prove that g is analytic, and then to construct an antiderivative of g, which will be almost the function f that we're after. To prove that g is analytic, we use Theorem 2.6: first because u is harmonic, $\text{Re } g = u_x$ and $\text{Im } g = -u_y$ have continuous partials. Moreover, again because u is harmonic, they satisfy the Cauchy–Riemann equations:

$$(\text{Re } g)_x = u_{xx} = -u_{yy} = (\text{Im } g)_y$$

and

$$(\text{Re } g)_y = u_{xy} = u_{yx} = -(\text{Im } g)_x \,.$$

Now that we know that g is analytic in G, we can use Theorem 5.6 to obtain a primitive h of g on G. (Note that for the application of this theorem we need the fact that G is simply connected.) Suppose we decompose h into its real and imaginary parts as $h = a + ib$. Then, again using Theorem 2.6,

$$g = h' = a_x + ib_x = a_x - ia_y \,.$$

(The second equation follows with the Cauchy–Riemann equations.) But the real part of g is u_x, so that we obtain $u_x = a_x$ or $u(x, y) = a(x, y) + c(y)$ for some function c which only depends on y. On the other hand, comparing the imaginary parts of g and h' yields $-u_y = -a_y$ or $u(x, y) = a(x, y) + c(x)$, and c depends only on x. Hence c has to be constant, and $u = a + c$. But then

$$f = h - c$$

is a function analytic in G whose real part is u, as promised. □

Remark. In hindsight, it should not be surprising that the function g which we first constructed is the derivative of the sought-after function f. Namely, by Theorem 2.6 such a function $f = u + iv$ must satisfy

$$f' = u_x + iv_x = u_x - iu_y \,.$$

(The second equation follows with the Cauchy–Riemann equations.) It is also worth mentioning that the proof shows that if u is harmonic in G then u_x is the real part of a function analytic in G *regardless whether G is simply connected or not.*

As one might imagine, the two theorems we've just proved allow for a powerful interplay between harmonic and analytic functions. In that spirit, the following theorem might appear not too surprising. It is, however, a very strong result, which one might appreciate better when looking back at the simple definition of harmonic functions.

Corollary 6.3. *A harmonic function is infinitely differentiable.*

Proof. Suppose u is harmonic in G. Fix $z_0 \in G$ and $r > 0$ such that the disk

$$D = \{z \in \mathbb{C} : |z - z_0| < r\}$$

is contained in G. D is simply connected, so by the last theorem, there exists a function f analytic in D such that $u = \operatorname{Re} f$ on D. By Corollary 5.2, f is infinitely differentiable on D, and hence so is its real part u. Because $z_0 \in D$, we showed that u is infinitely differentiable at z_0, and because z_0 was chosen arbitrarily, we proved the statement. □

Remark. This is the first in a series of proofs which uses the fact that the property of being harmonic is a *local* property—it is a property at each point of a certain region. Note that we did not construct a function f which is analytic in G but we only constructed such a function on the disk D. This f might very well differ from one disk to the next.

6.2 Mean-Value and Maximum/Minimum Principle

The following identity is the harmonic analog of Cauchy's integral formula, Theorem 4.8.

Theorem 6.4. *Suppose u is harmonic in the region G, and $\{z \in \mathbb{C} : |z - w| \leq r\} \subset G$. Then*

$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} u\left(w + re^{it}\right) dt \,.$$

Proof. The disk $D = \{z \in \mathbb{C} : |z - w| \leq r\}$ is simply connected, so by Theorem 6.2 there is a function f analytic on D such that $u = \operatorname{Re} f$ on D. Now we apply Corollary 4.10 to f:

$$f(w) = \frac{1}{2\pi} \int_0^{2\pi} f\left(w + re^{it}\right) dt \,.$$

The statement follows by taking the real part on both sides. □

Theorem 6.4 states that harmonic functions have the *mean-value property*. The following result is a fairly straightforward consequence of this property. The function $u : G \subset \mathbb{C} \to \mathbb{R}$ has a *strong relative maximum* at w if there exists a disk $D = \{z \in \mathbb{C} : |z - w| < R\} \subset G$ such that $u(z) \leq u(w)$ for all $z \in D$ and $u(z_0) < u(w)$ for some $z_0 \in D$. The definition of a *strong relative minimum* is completely analogous.

Theorem 6.5. *If u is harmonic in the region G, then it does not have a strong relative maximum or minimum in G.*

Proof. Assume (by way of contradiction) that w is a strong local maximum of u in G. Then there is a disk in G centered at w containing a point z_0 with $u(z_0) < u(w)$. Suppose $|z_0 - w| = r$; we apply Theorem 6.4 with this r:

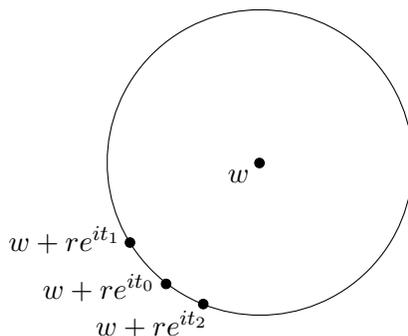$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} u\left(w + re^{it}\right) dt \,.$$

Figure 6.1: Proof of Theorem 6.5.

Intuitively, this cannot hold, because some of the function values we're integrating are smaller than $u(w)$, contradicting the mean-value property. To make this into a thorough argument, suppose that $z_0 = w + re^{it_0}$. Because $u(z_0) < u(w)$ and u is continuous, there is a whole interval of parameters, say $t_1 \le t \le t_2$ (and t_0 is among those t), such that $u\left(w + re^{it}\right) < u(w)$.

Now we split up the mean-value integral:

$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} u\left(w + re^{it}\right) dt$$

$$= \frac{1}{2\pi} \left(\int_0^{t_1} u\left(w + re^{it}\right) dt + \int_{t_1}^{t_2} u\left(w + re^{it}\right) dt + \int_{t_2}^{2\pi} u\left(w + re^{it}\right) dt \right)$$

All the integrands can be bounded by $u(w)$, for the middle integral we get a *strict* inequality. Hence

$$u(w) < \frac{1}{2\pi} \left(\int_0^{t_1} u(w)\, dt + \int_{t_1}^{t_2} u(w)\, dt \int_{t_2}^{2\pi} u(w)\, dt \right) = u(w)\,,$$

a contradiction. The same argument works if we assume that u has a relative minimum. But in this case there's actually a short cut: if u has a strong relative minimum then the harmonic function $-u$ has a strong relative maximum, which we just showed cannot exist. □

A look into the (not so distant) future. We will see in Corollary 8.12 a variation of this theorem for a *weak relative maximum* w, in the sense that there exists a disk $D = \{z \in \mathbb{C} : |z - w| < R\} \subset G$ such that all $z \in D$ satisfy $u(z) \le u(w)$. Corollary 8.12 says that if u is harmonic in the region G, then it does not have a *weak* relative maximum or minimum in G. A special yet important case of the above *maximum/minimum principle* is obtained when considering bounded regions. Corollary 8.12 implies that if u is harmonic in the closure of the bounded region G then

$$\max_{z \in G} u(z) = \max_{z \in \partial G} u(z) \qquad \text{and} \qquad \min_{z \in G} u(z) = \min_{z \in \partial G} u(z)\,.$$

(Here ∂G denotes the boundary of G.) We'll exploit this fact in the next two corollaries.

Corollary 6.6. *Suppose u is harmonic in the closure of the bounded region G. If u is zero on ∂G then u is zero in G.*

Proof. By the remark we just made

$$u(z) \le \max_{z \in G} u(z) = \max_{z \in \partial G} u(z) = \max_{z \in \partial G} 0 = 0$$

and

$$u(z) \ge \min_{z \in G} u(z) = \min_{z \in \partial G} u(z) = \min_{z \in \partial G} 0 = 0\,,$$

so u has to be zero in G. □

Corollary 6.7. *If two harmonic functions agree on the boundary of a bounded region then they agree in the region.*

Proof. Suppose u and v are harmonic in $G \cup \partial G$ and they agree on ∂G. Then $u-v$ is also harmonic in $G \cup \partial G$ (Exercise 2) and $u - v$ is zero on ∂G. Now apply the previous corollary. □

The last corollary states that if we know a harmonic function on the boundary of some region then we know it inside the region. One should remark, however, that this result is of a completely theoretical nature: it says nothing about how to extend a function given on the boundary of a region to the full region. This problem is called the *Dirichlet*[2] *problem* and has a solution for all simply-connected regions. There is a fairly simple formula (involving the so-called *Poisson*[3] *kernel*) if the region in question is a disk; for other regions one needs to find a *conformal map* to the unit disk. All of this is beyond the scope of these notes, we just remark that Corollary 6.7 says that the solution to the Dirichlet problem is unique.

Exercises

1. Show that all partial derivatives of a harmonic function are harmonic.

2. Suppose u and v are harmonic, and $c \in \mathbb{R}$. Prove that $u + cv$ is also harmonic.

3. Consider $u(z) = u(x, y) = \ln\left(x^2 + y^2\right)$.

   (a) Show that u is harmonic in $\mathbb{C} \setminus \{0\}$.

   (b) Prove that u is *not* the real part of a function which is analytic in $\mathbb{C} \setminus \{0\}$.

4. Let $u(x, y) = e^x \sin y$.

   (a) Show that u is harmonic on $\mathbb{C}$.

   (b) Find an entire function f such that $\text{Re}(f) = u$.

5. Is it possible to find a real function v so that $x^3 + y^3 + iv$ is analytic?

---

[2]For more information about Johann Peter Gustav Dirichlet (1805–1859), see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Dirichlet.html`.
[3]For more information about Siméon Denis Poisson (1781–1840), see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Poisson.html`.

Chapter 7

Power Series

It is a pain to think about convergence but sometimes you really have to.
Sinai Robins

7.1   Sequences and Completeness

As in the real case (and there will be no surprises in this chapter of the nature 'real versus complex'), a *(complex) sequence* is a function from the positive (sometimes the nonnegative) integers to the complex numbers. Its values are usually denoted by a_n (as opposed to, say, $a(n)$) and we commonly denote the sequence by $(a_n)_{n=1}^{\infty}$, $(a_n)_{n \geq 1}$, or simply (a_n). The notion of convergence of a sequence is based on the following sibling of Definition 2.1.

Definition 7.1. Suppose (a_n) is a sequence and $a \in \mathbb{C}$ such that for all $\epsilon > 0$, there is an integer N such that for all $n \geq N$, we have $|a_n - a| < \epsilon$. Then the sequence (a_n) is *convergent* and a is its *limit*, in symbols

$$\lim_{n \to \infty} a_n = a\,.$$

If no such a exists then the sequence (a_n) is *divergent*.

Example 7.1. $\lim_{n \to \infty} \frac{i^n}{n} = 0$: Given $\epsilon > 0$, choose $N > 1/\epsilon$. Then for any $n \geq N$,

$$\left| \frac{i^n}{n} - 0 \right| = \left| \frac{i^n}{n} \right| = \frac{|i|^n}{n} = \frac{1}{n} \leq \frac{1}{N} < \epsilon\,.$$

Example 7.2. The sequence $(a_n = i^n)$ diverges: Given $a \in \mathbb{C}$, choose $\epsilon = 1/2$. We consider two cases: If $\operatorname{Re} a \geq 0$, then for any N, choose $n \geq N$ such that $a_n = -1$. (This is always possible since $a_{4k+2} = i^{4k+2} = -1$ for any $k \geq 0$.) Then

$$|a - a_n| = |a + 1| \geq 1 > \frac{1}{2}\,.$$

If $\operatorname{Re} a < 0$, then for any N, choose $n \geq N$ such that $a_n = 1$. (This is always possible since $a_{4k} = i^{4k} = 1$ for any $k > 0$.) Then

$$|a - a_n| = |a - 1| \geq 1 > \frac{1}{2}\,.$$

The following limit laws are the relatives of the identities stated in Lemma 2.2.

Lemma 7.1. *Let (a_n) and (b_n) be convergent sequences and $c \in \mathbb{C}$.*

(a) $\lim\limits_{n\to\infty} a_n + c \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} (a_n + c\, b_n)\,.$

(b) $\lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} (a_n \cdot b_n)\,.$

(c) $\dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \lim\limits_{n\to\infty} \left(\dfrac{a_n}{b_n}\right)\,.$

In the quotient law we have to make sure we do not divide by zero. Moreover, if f is continuous at a then

$$\lim_{n\to\infty} f(a_n) = f(a) \qquad \text{if} \qquad \lim_{n\to\infty} a_n = a\,,$$

where we require that a_n be in the domain of f.

The most important property of the real number system is that we can, in many cases, determine that a sequence converges *without knowing the value of the limit*. In this sense we can use the sequence to define a real number. In fact, all irrational numbers are actually defined this way, as limits of rational numbers. This property of the real numbers is called *completeness*, and it can be formulated in many equivalent ways. We will accept the following axiom as our version of the completeness property:

Axiom (Monotone Sequence Property). *Any bounded monotone sequence converges.*

Remember that a sequence is monotone if it is either non-decreasing $(x_{n+1} \geq x_n)$ or non-increasing $(x_{n+1} \leq x_n)$.

Example 7.3. If $0 \leq r < 1$ then $\lim_{n\to\infty} r^n = 0$: First, the sequence converges because it is decreasing and bounded below by 0. If the limit is L then, using the laws of limits, we get $L = \lim_{n\to\infty} r^n = \lim_{n\to\infty} r^{n+1} = r \lim_{n\to\infty} r^n = rL$. From $L = rL$ we get $(1 - r)L = 0$, so $L = 0$ since $1 - r \neq 0$

The following is a consequence of the monotone sequence property, although it is often listed as a separate axiom:

Theorem 7.2 (Archimedean Property). *If x is any real number than there is an integer N which is greater than x.*

This essentially says that there are no infinities in the reals. Notice that this was already used in Example 7.1. For a proof see Exercise 3. It is interesting to see that the Archimedean principle underlies the construction of an infinite decimal expansion for any real number, while the monotone sequence property shows that any such infinite decimal expansion actually converges to a real number.

We close this discussion of limits with a pair of standard limits. The first of these can be established by calculus methods (like L'Hospital's rule, by treating n as the variable); both of them can be proved by more elementary considerations.

Lemma 7.3. (a) *Exponentials beat polynomials:* $\lim_{n\to\infty} b^n p(n) = 0$ *if $p(n)$ is a polynomial of fixed degree in n and $|b| < 1$.*

(b) *Factorials beat exponentials:* $\lim_{n\to\infty} \dfrac{a^n}{n!} = 0$ *if a is a constant.*

7.2   Series

A *series* is a sequence (a_n) whose members are of the form $a_n = \sum_{k=1}^{n} b_k$ (or $a_n = \sum_{k=0}^{n} b_k$); here (b_k) is the *sequence of terms* of the series. The $a_n = \sum_{k=1}^{n} b_k$ (or $a_n = \sum_{k=0}^{n} b_k$) are the *partial sums* of the series. If we wanted to be lazy we would for convergence of a series simply refer to convergence of the partial sums of the series, after all, we just defined series through sequences. However, there are some convergence features which take on special appearances for series, so we should mention them here explicitly. For starters, a series *converges* to the *limit* (or *sum*) a by definition if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} b_k = a\,.$$

To express this in terms of Definition 7.1, for any $\epsilon > 0$ we have to find an N such that for all $n \geq N$

$$\left| \sum_{k=1}^{n} b_k - a \right| < \epsilon\,.$$

In the case of a convergent series, we usually express its limit as $a = \sum_{k=1}^{\infty} b_k$ or $a = \sum_{k\geq 1} b_k$.

Example 7.4. Occasionally we can find the limit of a sequence by manipulating the partial sums:

$$\begin{aligned}
\sum_{k\geq 1} \frac{1}{k(k+1)} &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&= \lim_{n\to\infty} \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \right] \\
&= \lim_{n\to\infty} \left[1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1} \right] \\
&= \lim_{n\to\infty} \left[1 - \frac{1}{n+1} \right] = 1.
\end{aligned}$$

A series where most of the terms cancel like this is called a *telescoping series.*

Most of the time we need to use the completeness property to check convergence of a series, and it is fortunate that the monotone sequence property has a very convenient translation into the language of series of real numbers. The partial sums of a series form a nondecreasing sequence if the terms of the series are nonnegative, and this observation immediately yields:

Lemma 7.4. *If b_k are nonnegative real numbers then $\sum_{k=1}^{\infty} b_k$ converges if and only if the partial sums are bounded.*

If b_k are nonnegative real numbers and the partial sums of the series $\sum_{k=1}^{\infty} b_k$ are unbounded then the partial sums "converge" to infinity, so we can write $\sum_{k=1}^{\infty} b_k = \infty$. Using this terminology, we can rephrase Lemma 7.4 to say: $\sum_{k=1}^{\infty} b_k$ converges in the reals if and only if it is finite.

We have already used the simple fact that convergence of a sequence (a_n) is equivalent to the convergence of (a_{n-1}), and both of these sequences have the same limit. If a_n is the nth partial sum of the series $\sum_{k\geq 1} b_k$ then $a_n = a_{n-1} + b_n$. From this we conclude:

Lemma 7.5. *If $\sum_{k\geq 1} b_k$ converges then $\lim_{n\to\infty} b_n = 0$.*

A common mistake is to try to use the converse of this result, but the converse is false:

Example 7.5. The *harmonic series* $\sum_{k \geq 1} \frac{1}{k}$ diverges (even though the limit of the general term is 0): If we assume the series converges, say to L, then we have

$$
\begin{aligned}
L &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \\
&= \left(1 + \frac{1}{3} + \frac{1}{5} + \dots \right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots \right) \\
&> \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots \right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots \right) \\
&= \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \right) + \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \right) \\
&= \frac{1}{2} L + \frac{1}{2} L = L.
\end{aligned}
$$

Here the inequality comes from $\frac{1}{k} > \frac{1}{k+1}$ applied to each term in the first sum in parentheses. But now we have $L > L$, which is impossible.

There is one notion of convergence that's special to series: we say that $\sum_{k \geq 1} c_k$ *converges absolutely* if $\sum_{k \geq 1} |c_k| < \infty$. Be careful: We are defining the *phrase* "converges absolutely," but this definition does not say anything about convergence of the series $\sum_{k \geq 1} c_k$; we need a proof:

Theorem 7.6. *If a series converges absolutely then it converges.*

Proof. First consider the case when the terms c_k are real. Define c_k^+ to be c_k if $c_k \geq 0$, or 0 if $c_k < 0$. Then $c_k^+ \geq 0$ and $\sum_{k \geq 1} c_k^+ \leq \sum_{k \geq 1} |c_k| < \infty$ so $\sum_{k \geq 1} c_k^+$ converges; let P be its limit. Similarly, define c_k^- to be $-c_k$ if $c_k \leq 0$, or 0 if $c_k > 0$. Then $c_k^- \geq 0$ and $\sum_{k \geq 1} c_k^- \leq \sum_{k \geq 1} |c_k| < \infty$ so $\sum_{k \geq 1} c_k^-$ converges; let N be its limit. Since $c_k = c_k^+ - c_k^-$ we see that $\sum_{k \geq 1} c_k$ converges to $P - N$.

In case c_k is complex, write $c_k = a_k + ib_k$ where a_k and b_k are real. Then $\sum_{k \geq 1} |a_k| \leq \sum_{k \geq 1} |c_k| < \infty$ and $\sum_{k \geq 1} |b_k| \leq \sum_{k \geq 1} |c_k| < \infty$. By what we just proved, both $\sum_{k \geq 1} a_k$ and $\sum_{k \geq 1} b_k$ converge to real numbers, say, A and B. But then $\sum_{k \geq 1} c_k$ converges to $A + iB$. $\square$

Another common mistake is to try to use the converse of this result, but the converse is false:

Example 7.6. The *alternating harmonic series* $\sum_{k \geq 1} \frac{(-1)^{k+1}}{k}$ converges, but not absolutely: This series does not converge absolutely, according to the previous example. To see that it does converge, rewrite it as follows:

$$
\begin{aligned}
\sum_{k \geq 1} \frac{(-1)^{k+1}}{k} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \\
&= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) + \dots
\end{aligned}
$$

(Technically, there is a small detail to be checked here, since we are effectively ignoring half the partial sums of the original series. See Exercise 10.) The reader can verify the inequality $2k(2k-1) \geq k(k+1)$ for $k > 1$, so the general term satisfies

$$
\frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{2k(2k-1)} \leq \frac{1}{k(k+1)},
$$

so the series converges by comparison with the telescoping series of Example 7.4.

For the rest of this book we shall be concerned almost exclusively with series which converge absolutely. Hence checking convergence of a series is usually a matter of verifying that a series of nonnegative reals is finite. We have already used the technique of comparing a series to a series which is known to converge; this is often called a "comparison test." Some variants of the comparison test will appear when we look at power series. One handy test is the following:

Lemma 7.7 (Integral Test). *Suppose* f *is a non-increasing, positive function defined on* $[1, \infty)$. *Then*

$$\int_1^\infty f(t)\, dt \leq \sum_{k=1}^\infty f(k) \leq f(1) + \int_1^\infty f(t)\, dt$$

This is immediate from a picture: the integral of $f(t)$ on the interval $[k, k+1]$ is bounded between $f(k)$ and $f(k+1)$. Adding the pieces gives the inequalities above for the Nth partial sum versus the integrals from 1 to N and from 1 to $N+1$, and the inequality persists in the limit.

Example 7.7. $\sum_{k \geq 1} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

7.3 Sequences and Series of Functions

The fun starts when one studies sequences (f_n) of functions f_n. We say that such a sequence *converges at* z_0 if the sequence (of complex numbers) $(f_n(z_0))$ converges. If a sequence of functions, (f_n), converges at all z in some subset $G \subseteq \mathbb{C}$ then we say that (f_n) *converges pointwise on* G. So far nothing new; but this notion of convergence does not really catch the spirit of the function as a whole.

Definition 7.2. Suppose (f_n) and f are functions defined on $G \subseteq \mathbb{C}$. If for all $\epsilon > 0$ there is an N such that for all $z \in G$ and for all $n \geq N$ we have

$$|f_n(z) - f(z)| < \epsilon$$

then (f_n) *converges uniformly in* G *to* f.

What's the big deal about uniform versus pointwise convergence? It is easiest to describe the difference with the use of quantifiers, namely $\forall$ denoting "for all" and $\exists$ denoting "there is." Pointwise convergence on G means

$$(\forall \epsilon > 0)\,(\forall z \in G)\,(\exists N : n \geq N \Rightarrow |f_n(z) - f(z)| < \epsilon)\,,$$

whereas uniform convergence on G translates into

$$(\forall \epsilon > 0)\,(\exists N : (\forall z \in G)\ n \geq N \Rightarrow |f_n(z) - f(z)| < \epsilon)\,.$$

No big deal—we only exchanged two of the quantifiers. In the first case, N may well depend on z, in the second case we need to find an N which works for all $z \in G$. And this can make all the difference . . .

The first example illustrating this difference says in essence that if we have a sequence of functions (f_n) which converges uniformly on G then for all $z_0 \in G$

$$\lim_{n \to \infty} \lim_{z \to z_0} f_n(z) = \lim_{z \to z_0} \lim_{n \to \infty} f_n(z)\,.$$

We will need similar interchanges of limits constantly.

Proposition 7.8. *Suppose* (f_n) *is a sequence of continuous functions on the region* G *converging uniformly to* f *on* G*. Then* f *is continuous on* G*.*

Proof. Let $z_0 \in G$; we will prove that f is continuous at z_0. By uniform convergence, given $\epsilon > 0$, there is an N such that for all $z \in G$ and all $n \geq N$

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}.$$

Now we make use of the continuity of the f_n's. This means that given (the same) $\epsilon > 0$, there is a $\delta > 0$ such that whenever $|z - z_0| < \delta$ we have

$$|f_n(z) - f_n(z_0)| < \frac{\epsilon}{3}.$$

All that's left is putting those two inequalities together: by the triangle inequality

$$
\begin{aligned}
|f(z) - f(z_0)| &= |f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0)| \\
&\leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| \\
&< \epsilon,
\end{aligned}
$$

that is, f is continuous at z_0. $\qquad\square$

Once we know the above result about continuity, we can ask about integration of series of functions. The next theorem should come as no surprise, however, its consequences (which we will only see in the next chapter) are wide ranging.

Proposition 7.9. *Suppose* f_n *are continuous on the smooth curve* γ *and converge uniformly on* γ *to* f*. Then*

$$\lim_{n \to \infty} \int_\gamma f_n = \int_\gamma f.$$

Proof. By Proposition 4.1(d), we can estimate

$$\left| \int_\gamma f_n - \int_\gamma f \right| = \left| \int_\gamma f_n - f \right| \leq \max_{z \in \gamma} |f_n(z) - f(z)| \operatorname{length}(\gamma).$$

But $f_n \to f$ uniformly on γ, and we can make $\max_{z \in \gamma} |f_n(z) - f(z)|$ as small as we like. $\qquad\square$

Since uniform convergence is often of critical importance, we give two practical tests: one arguing for uniformity and the other against. They are formulated for sequences that converge to 0. If a sequence g_n converges to a function g then we can usually apply these tests to $f_n = g - g_n$, which does converge to 0.

Lemma 7.10. *If* f_n *is a sequence of functions and* M_n *is a sequence of constants so that* M_n *converges to* 0 *and* $|f_n(z)| \leq M_n$ *for all* $z \in G$*, then* f_n *converges uniformly to* 0 *on* G*.*

For example, $|z^n| \leq r^n$ if z is in the closed disk $\bar{D}_r(0)$, and $r^n \to 0$ if $r < 1$, so $z^n \to 0$ uniformly in $\bar{D}_r(0)$ if $r < 1$.

Lemma 7.11. *If* f_n *is a sequence of functions which converges uniformly to* 0 *on a set* G *and* z_n *is any sequence in* G *then the sequence* $f_n(z_n)$ *converges to* 0*.*

This is most often used to prove *non-uniform* convergence. For example, let $f_n(z) = z^n$ and let G be the open unit disk $D_1(0)$. Then $|z| < 1$ if z is in G, so $|z|^n \to 0$, and so $z^n \to 0$. However, let $z_n = \exp(-\frac{1}{n})$. Then z_n is in G but $f_n(z_n) = e^{-1}$ so $f_n(z_n)$ does not converge to 0. Therefore z^n does not converge uniformly to 0 on $D_1(0)$.

All of these notions for sequences of functions go verbatim for series of functions. Here we also have a notion of absolute convergence (which can be combined with uniform convergence). There is an important result about series of functions, often called the *Weierstraß M-test*.

Proposition 7.12. *Suppose (f_k) are continuous on the region G, $|f_k(z)| \le M_k$ for all $z \in G$, and $\sum_{k \ge 1} M_k$ converges. Then $\sum_{k \ge 1} f_k$ converges absolutely and uniformly in G.*

Proof. For each fixed z we have $\sum_{k \ge 1} |f_k(z)| \le \sum_{k \ge 1} M_k < \infty$, so $\sum_{k \ge 1} f_k(z)$ converges; call the limit $f(z)$. This defines a function f on G. To see that f_n converges uniformly to f, suppose $\epsilon > 0$. Since $\sum_{k \ge 1} M_k$ converges there is N so that

$$\sum_{k > n} M_k = \sum_{k=1}^{\infty} M_k - \sum_{k=1}^{n} M_k < \epsilon$$

for all $n > N$. Then, for any z in G, if $n \ge N$ then

$$\left| f(z) - \sum_{k=1}^{n} f_k(z) \right| = \left| \sum_{k > n} f_n(z) \right| \le \sum_{k > n} |f_n(z)| \le \sum_{k > n} M_k < \epsilon$$

and this satisfies the definition of uniform convergence. $\square$

7.4 Region of Convergence

For the remainder of this chapter (indeed, these lecture notes) we concentrate on some very special series of functions.

Definition 7.3. A *power series centered at z_0* is a series of functions of the form

$$\sum_{k \ge 0} c_k (z - z_0)^k .$$

The fundamental example of a power series is the *geometric series*, for which all $c_k = 1$.

Lemma 7.13. *The geometric series $\sum_{k \ge 0} z^k$ converges absolutely for $|z| < 1$ to the function $1/(1 - z)$. The convergence is uniform on any set of the form $\{\, z \in \mathbb{C} : |z| \le r \,\}$ for any $r < 1$.*

Proof. Fix an $r < 1$, and let $D = \{\, z \in \mathbb{C} : |z| \le r \,\}$. We will use Proposition 7.12 with $f_k(z) = z^k$ and $M_k = r^k$. Hence the uniform convergence on D of the geometric series will follow if we can show that $\sum_{k \ge 0} r^k$ converges. But this is straightforward: the partial sums of this series can be written as

$$\sum_{k=0}^{n} r^k = 1 + r + \cdots + r^{n-1} + r^n = \frac{1 - r^{n+1}}{1 - r} ,$$

whose limit as $n \to \infty$ exists because $r < 1$. Hence, by Proposition 7.12, the geometric series converges absolutely and uniformly on any set of the form $\{z \in \mathbb{C} : |z| \leq r\}$ with $r < 1$. Since r can be chosen arbitrarily close to 1, we have absolute convergence for $|z| < 1$. It remains to show that for those z the limit function is $1/(1-z)$, which follows by

$$\sum_{k \geq 0} z^k = \lim_{n \to \infty} \sum_{k=0}^{n} z^k = \lim_{n \to \infty} \frac{1 - z^{n+1}}{1-z} = \frac{1}{1-z} . \qquad \square$$

By comparing a general power series to a geometric series we can give a complete description of its region of convergence.

Theorem 7.14. *Any power series $\sum_{k \geq 0} c_k (z - z_0)^k$ has a* radius of convergence R. *By this we mean that R is a nonnegative real number, or ∞, satisfying the following.*

(a) *If $r < R$ then $\sum_{k \geq 0} c_k (z - z_0)^k$ converges absolutely and uniformly on the closed disk $\bar{D}_r(z_0)$ of radius r centered at z_0.*

(b) *If $|z - z_0| > R$ then the sequence of terms $c_k (z - z_0)^k$ is unbounded, so $\sum_{k \geq 0} c_k (z - z_0)^k$ does not converge.*

The open disk $D_R(z_0)$ in which the power series converges absolutely is the *region of convergence*. (If $R = \infty$ then $D_R(z_0)$ is the entire complex plane, and if $R = 0$ then $D_R(z_0)$ is the empty set.) By way of Proposition 7.8, this theorem immediately implies the following.

Corollary 7.15. *Suppose the power series $\sum_{k \geq 0} c_k (z - z_0)^k$ has radius of convergence R. Then the series represents a function which is continuous on $D_R(z_0)$.*

While we're at it, we might as well state what Proposition 7.9 implies for power series.

Corollary 7.16. *Suppose the power series $\sum_{k \geq 0} c_k (z - z_0)^k$ has radius of convergence R and γ is a smooth curve in $D_R(z_0)$. Then*

$$\int_\gamma \sum_{k \geq 0} c_k (z - z_0)^k \, dz = \sum_{k \geq 0} c_k \int_\gamma (z - z_0)^k \, dz .$$

In particular, if γ is closed then $\int_\gamma \sum_{k \geq 0} c_k (z - z_0)^k \, dz = 0$.

Proof of Theorem 7.14. Define C to be the set of positive real numbers for which the series $\sum_{k \geq 0} c_k t^k$ converges, and define D to be the set of positive real numbers for which it diverges. Clearly every positive real number is in either C or D, and these sets are disjoint. First we establish three facts about these sets.

$(*)$ If $r < t$ and $t \in C$ then $r \in C$ and $\sum_{k \geq 0} c_k (z - z_0)^k$ converges absolutely and uniformly on $\bar{D}_r(z_0)$. To prove this, note that $\sum_{k \geq 0} c_k t^k$ converges so $c_k t^k \to 0$ as $k \to \infty$. In particular, this sequence is bounded, so $|c_k| t^k \leq M$ for some constant M. Now if $z \in \bar{D}_r(z_0)$ we have $\left| c_k (z - z_0)^k \right| \leq |c_k| r^k$ and

$$\sum_{k \geq 0} |c_k| r^k = \sum_{k \geq 0} |c_k| t^k \left(\frac{r}{t}\right)^k \leq \sum_{k \geq 0} M \left(\frac{r}{t}\right)^k = M \sum_{k \geq 0} \left(\frac{r}{t}\right)^k = \frac{M}{1 - r/t} < \infty.$$

At the last step we recognized the geometric series, which converges since $0 \leq r < t$, and so $0 \leq r/t < 1$. This shows that $r \in C$, and uniform and absolute convergence on $\bar{D}_r(z_0)$ follows from the Weierstraß M-test.

($**$) If $|z - z_0| = r > t$ and $t \in D$ then $r \in D$ and the sequence $c_k r^k$ is unbounded, and hence $\sum_{k \geq 0} c_k (z - z_0)^k$ diverges. To prove this, assume that $c_k r^k$ is bounded, so $|c_k| r^k \leq M$ for some constant M. But now exactly the same argument as in ($*$), but interchanging r and t, shows that $\sum_{k \geq 0} c_k t^k$ converges, contradicting the assumption that t is in D.

($* * *$) There is an extended real number R, satisfying $0 \leq R \leq \infty$, so that $0 < r < R$ implies $r \in C$ and $R < r < \infty$ implies $r \in D$. Notice that $R = 0$ works if C is empty, and $R = \infty$ works if D is empty; so we assume neither is empty and we start with a_0 in C and b_0 in D. It is immediate from ($*$) or ($**$) that $a_0 < b_0$. We shall define sequences a_n in C and b_n in D which "zero in" on R. First, let m_0 be the midpoint of the segment $[a_0, b_0]$, so $m_0 = (a_0 + b_0)/2$. If m_0 lies in C then we define $a_1 = m_0$ and $b_1 = b_0$; but if m_0 lies in D then we define $a_1 = a_0$ and $b_1 = m_0$. Note that, in either case, we have $a_0 \leq a_1 < b_1 \leq b_0$, a_1 is in C, and b_1 is in D. Moreover, a_1 and b_1 are closer together than a_0 and b_0; in fact, $b_1 - a_1 = (b_0 - a_0)/2$. We repeat this procedure to define a_2 and b_2 within the interval $[a_1, b_1]$, and so on. Summarizing, we have

$$a_n \leq a_{n+1} \qquad a_n \in C$$
$$b_n \geq b_{n+1} \qquad b_n \in D$$
$$a_n < b_n$$
$$b_n - a_n = (b_0 - a_0)/2^n$$

The sequences a_n and b_n are monotone and bounded (by a_0 and b_0) so they have limits, and these limits are the same since $\lim_{n \to \infty}(b_n - a_n) = \lim_{n \to \infty}(b_0 - a_0)/2^n = 0$. We define R to be this limit. If $0 < r < R$ then $r < a_n$ for all sufficiently large n, since a_n converges to R, so r is in C by ($*$). On the other hand, if $R < r$ then $b_n < r$ for all sufficiently large n, so r is in D by ($**$). Thus R verifies the statement ($* * *$).

To prove Theorem 7.14, first assume $r < R$ and choose t so that $r < t < R$. Then $t \in C$ by ($* * *$), so part (a) of 7.14 follows from ($*$). Similarly, if $r = |z - z_0| > R$ then choose t so that $R < t < r$. Then $t \in D$ by ($* * *$), so part (b) of 7.14 follows from ($**$). $\qquad \square$

It is worth mentioning the following corollary, which reduces the calculation of the radius of convergence to examining the limiting behavior of the terms of the series.

Corollary 7.17. $|c_k| r^k \to 0$ *for* $0 \leq r < R$ *but* $|c_k| r^k$ *is unbounded for* $r > R$.

Warning: Neither Theorem 7.14 nor Corollary 7.17 says anything about convergence on the circle $|z - z_0| = R$.

Exercises

1. For each of the sequences, prove convergence/divergence. If the sequence converges, find the limit.

   (a) $a_n = e^{i\pi n/4}$.
   (b) $\frac{(-1)^n}{n}$.

    (c) $\cos n$.

    (d) $2 - \frac{in^2}{2n^2+1}$.

    (e) $\sin\left(\frac{1}{n}\right)$.

2. Show that the limit of a convergent sequence is unique.

3. Derive the Archimedean Property from the monotone sequence property.

4. Prove:

    (a) $\lim\limits_{n\to\infty} a_n = a \quad\Longrightarrow\quad \lim\limits_{n\to\infty} |a_n| = |a|$.

    (b) $\lim\limits_{n\to\infty} a_n = 0 \quad\Longleftrightarrow\quad \lim\limits_{n\to\infty} |a_n| = 0$.

5. Prove Lemma 7.3.

6. Prove: (c_n) converges if and only if $(\mathrm{Re}\, c_n)$ and $(\mathrm{Im}\, c_n)$ converge.

7. Prove Lemma 7.1.

8. Suppose $a_n \le b_n \le c_n$ for all n and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$. Prove that $\lim_{n\to\infty} b_n = L$. State and prove a similar theorem for series.

9. Find $\sup\left\{\mathrm{Re}\left(e^{2\pi it}\right) : t \in \mathbb{Q} \setminus \mathbb{Z}\right\}$.

10. Suppose that the terms c_n converge to zero, and show that $\sum_{n=0}^{\infty} c_n$ converges if and only if $\sum_{k=0}^{\infty}(c_{2k} + c_{2k+1})$ converges. Moreover, if the two series converge then they have the same limit. Also, give an example where c_n does not converge to 0 and one series diverges while the other converges.

11. Prove that the series $\sum_{k\ge 1} b_k$ converges if and only if $\lim_{n\to\infty} \sum_{k=n}^{\infty} b_k = 0$.

12.   (a) Show that $\sum_{k\ge 1} \frac{1}{2^k} = 1$. One way to do this is to write $\frac{1}{2^k}$ as a difference of powers of 2 so that you get a telescoping series.

    (b) Show that $\sum_{k\ge 1} \frac{k}{k^2+1}$ diverges. (*Hint*: compare the general term to $\frac{1}{2k}$.)

    (c) Show that $\sum_{k\ge 1} \frac{k}{k^3+1}$ converges. (*Hint*: compare the general term to $\frac{1}{k^2}$.)

13. Discuss the convergence of $\sum_{k\ge 0} z^k$ for $|z| = 1$.

14. Prove Lemma 7.10.

15. Prove Lemma 7.11.

16. Discuss pointwise and uniform convergence for the following sequences

    (a) (nz^n).

    (b) $\left(\frac{z^n}{n}\right)$ for $n > 0$.

    (c) $\left(\frac{1}{1+nz}\right)$, defined on $\{z \in \mathbb{C} : \mathrm{Re}\, z \ge 0\}$.

17. Let $f_n(x) = n^2 x e^{-nx}$.

    (a) Show that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \geq 0$. Treat $x = 0$ as a special case; for $x > 0$ you can use L'Hospital's rule—but remember that n is the variable, not x.

    (b) Find $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$. (*Hint*: the answer is *not* 0.)

    (c) Why doesn't your answer to part (b) violate Proposition 7.9?

18. Find a power series (and determine its radius of convergence) of the following functions.

    (a) $\frac{1}{1+4z}$.

    (b) $\frac{1}{3-\frac{z}{2}}$.

19. (a) Suppose that the sequence c_k is bounded and show that the radius of convergence of $\sum_{k\geq 0} c_k(z - z_0)^k$ is at least 1.

    (b) Suppose that the sequence c_k does not converge to 0 and show that the radius of convergence of $\sum_{k\geq 0} c_k(z - z_0)^k$ is at most 1.

20. Find the power series centered at 1 for the following functions, and compute their radius of convergence:

    (a) $\frac{1}{z}$.

    (b) $\mathrm{Log}\, z$.

21. Use the Weierstraß M-test to show that each of the following series converges uniformly on the given domain:

    (a) $\displaystyle\sum_{k\geq 1} \frac{z^k}{k^2}$ on $\bar{D}_1(0)$.

    (b) $\displaystyle\sum_{k\geq 0} \frac{1}{z^k}$ on $\{z\colon\ |z| \geq 2\}$.

    (c) $\displaystyle\sum_{k\geq 0} \frac{z^k}{z^k + 1}$ on $\bar{D}_r(0)$, where $0 \leq r < 1$.

22. Suppose $L = \lim_{k\to\infty} |c_k|^{1/k}$ exists. Show that $\frac{1}{L}$ is the radius of convergence of $\sum_{k\geq 0} c_k\,(z - z_0)^k$. (Use the natural interpretations if $L = 0$ or $L = \infty$.)

23. Find the radius of convergence for each of the following series.

    (a) $\displaystyle\sum_{k\geq 0} a^{k^2} z^k$, $a \in \mathbb{C}$.

    (b) $\displaystyle\sum_{k\geq 0} k^n z^k$, $n \in \mathbb{Z}$.

    (c) $\displaystyle\sum_{k\geq 0} z^{k!}$.

(d) $\displaystyle\sum_{k\geq 1} \frac{(-1)^k}{k} z^{k(k+1)}$.

(e) $\displaystyle\sum_{k\geq 1} \frac{z^k}{k^k}$.

(f) $\displaystyle\sum_{k\geq 0} \cos(k) z^k$.

(g) $\displaystyle\sum_{k\geq 0} 4^k (z-2)^k$.

24. Define the functions $f_n(t) = \frac{1}{n} e^{-t/n}$ for $n > 0$ and $0 \leq t < \infty$.

   (a) Show that the maximum of $f_n(t)$ is $\frac{1}{n}$.

   (b) Show that $f_n(t)$ converges uniformly to 0 as $n \to \infty$.

   (c) Show that $\int_0^\infty f_n(t)\,dt$ does not converge to 0 as $n \to \infty$

   (d) Why doesn't this contradict the theorem that "the integral of a uniform limit is the limit of the integrals"?

25. Let f be analytic on the disk $|z| < 4$ and suppose $|f(z)| \leq 5$ for all z on the circle $|z| = 3$. Show that $\left|f^{(3)}(0)\right| \leq \frac{10}{9}$. (*Hint*: Use the Cauchy integral formula.) What can you say about $\left|f^{(3)}(1)\right|$?

Chapter 8

Taylor and Laurent Series

> *We think in generalities, but we live in details.*
> A. N. Whitehead

8.1 Power Series and Analytic Functions

All of the last chapter could have been developed in greater generality, say for functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will now, however, connect the last chapter to the theory of functions analytic on certain regions. The cornerstone is provided by two theorems which say that any power series represents an analytic function, and conversely, any analytic function can be represented as a power series. The first of them goes as follows.

Theorem 8.1. *Suppose $f(z) = \sum_{k \geq 0} c_k (z - z_0)^k$ has radius of convergence R. Then f is analytic in $\{z \in \mathbb{C} : |z - z_0| < R\}$.*

Proof. Given any closed curve $\gamma \subset \{z \in \mathbb{C} : |z - z_0| < R\}$, we have by Corollary 7.16

$$\int_\gamma \sum_{k \geq 0} c_k (z - z_0)^k \, dz = 0 \,.$$

On the other hand, Corollary 7.15 says that f is continuous. Now apply Morera's theorem (Corollary 5.7). □

A special case of the last result concerns power series with infinite radius of convergence: those represent entire functions.

Now that we know that power series are analytic (i.e., differentiable) on their regions of convergence we can ask how to find their derivatives. The next result says that we can simply differentiate the series "term by term."

Theorem 8.2. *Suppose $f(z) = \sum_{k \geq 0} c_k (z - z_0)^k$ has radius of convergence R. Then*

$$f'(z) = \sum_{k \geq 1} k \, c_k (z - z_0)^{k-1} \,,$$

and the radius of convergence of this power series is also R.

Proof. Let $f(z) = \sum_{k \geq 0} c_k (z - z_0)^k$. Since we know that f is analytic in its region of convergence we can use Theorem 5.1. Let γ be any simple closed curve in $\{z \in \mathbb{C} : |z - z_0| < R\}$. Note that the power series of f converges uniformly on γ, so that we are free to interchange integral and infinite sum. And then we use Theorem 5.1 *again*, but applied to the function $(z - z_0)^k$. Here are the details:

$$
\begin{aligned}
f'(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)^2} \, dw \\
&= \frac{1}{2\pi i} \int_\gamma \frac{\sum_{k \geq 0} c_k (w - z_0)^k}{(w - z)^2} \, dw \\
&= \sum_{k \geq 0} c_k \cdot \frac{1}{2\pi i} \int_\gamma \frac{(w - z_0)^k}{(w - z)^2} \, dw \\
&= \sum_{k \geq 0} c_k \cdot \frac{d}{dw} (w - z_0)^k \Big|_{w = z} \\
&= \sum_{k \geq 0} k \, c_k (z - z_0)^{k-1}.
\end{aligned}
$$

The last statement of the theorem is easy to show: the radius of convergence R of $f'(z)$ is at least R (since we have shown that the series converges whenever $|z - z_0| < R$), and it cannot be larger than R by comparison to the series for $f(z)$, since the coefficients for $(z - z_0)f'(z)$ are bigger than the corresponding ones for $f(z)$. $\square$

Naturally, the last theorem can be repeatedly applied to f', then to f'', and so on. The various derivatives of a power series can also be seen as ingredients of the series itself. This is the statement of the following *Taylor*[1] *series expansion.*

Corollary 8.3. *Suppose* $f(z) = \sum_{k \geq 0} c_k (z - z_0)^k$ *has a positive radius of convergence. Then*

$$
c_k = \frac{f^{(k)}(z_0)}{k!} .
$$

Proof. For starters, $f(z_0) = c_0$. Theorem 8.2 gives $f'(z_0) = c_1$. Applying the same theorem to f' gives

$$
f''(z) = \sum_{k \geq 2} k(k - 1)c_k (z - z_0)^{k-2}
$$

and $f''(z_0) = 2c_2$. We can play the same game for $f'''(z_0)$, $f''''(z_0)$, etc. $\square$

Taylor's formulas show that the coefficients of any power series which converges to f on an open disk D centered at z_0 can be determined from the the function f restricted to D. It follows immediately that the coefficients of a power series are unique:

Corollary 8.4 (Uniqueness of power series). *If* $\sum_{k \geq 0} c_k (z - z_0)^k$ *and* $\sum_{k \geq 0} c'_k (z - z_0)^k$ *are two power series which both converge to the same function* $f(z)$ *on an open disk centered at* z_0 *then* $c_k = c'_k$ *for all* k.

---

[1]For more information about Brook Taylor (1685–1731), see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Taylor.html`.

Theorem 8.5. *Suppose f is a function which is analytic in $D = \{z \in \mathbb{C} : |z - z_0| < R\}$. Then f can be represented in D as a power series centered at z_0 (with a radius of convergence at least R):*

$$f(z) = \sum_{k \geq 0} c_k \, (z - z_0)^k \qquad with \qquad c_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}} \, dw \,.$$

Here γ is any positively oriented, simple, closed, smooth curve in D for which z_0 is inside γ.

Proof. Let $g(z) = f(z + z_0)$; so g is a function analytic in $\{z \in \mathbb{C} : |z| < R\}$. Fix $r < R$, denote the circle centered at the origin with radius r by γ_r, and suppose that $|z| < r$. Then by Cauchy's integral formula (Theorem 4.8),

$$g(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{g(w)}{w - z} \, dw \,.$$

The factor $1/(w - z)$ in this integral can be extended into a geometric series (note that $w \in \gamma_r$ and so $\left|\frac{z}{w}\right| < 1$)

$$\frac{1}{w - z} = \frac{1}{w} \frac{1}{1 - \frac{z}{w}} = \frac{1}{w} \sum_{k \geq 0} \left(\frac{z}{w}\right)^k$$

which converges uniformly in the variable $w \in \gamma_r$ (by Lemma 7.13). Hence Proposition 7.9 applies:

$$g(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{g(w)}{w - z} \, dw = \frac{1}{2\pi i} \int_{\gamma_r} g(w) \frac{1}{w} \sum_{k \geq 0} \left(\frac{z}{w}\right)^k dw = \sum_{k \geq 0} \frac{1}{2\pi i} \int_{\gamma_r} \frac{g(w)}{w^{k+1}} \, dw \, z^k.$$

Now, since $f(z) = g(z - z_0)$, we apply an easy change of variables to obtain

$$f(z) = \sum_{k \geq 0} \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(w)}{(w - z_0)^{k+1}} \, dw \, (z - z_0)^k,$$

where Γ_r is a circle centered at z_0 with radius r. The only difference of this right-hand side to the statement of the theorem are the curves we're integrating over. However, $\Gamma_r \sim_{G \backslash \{z_0\}} \gamma$, and we can apply Cauchy's Theorem 4.4:

$$\int_{\Gamma_r} \frac{f(w)}{(w - z_0)^{k+1}} \, dw = \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}} \, dw \,. \qquad \square$$

If we compare the coefficients of the power series obtained in Theorem 8.5 with those in Corollary 8.3, we arrive at the long-promised extension of Theorem 5.1 (which in itself extended Cauchy's integral formula, Theorem 4.8).

Corollary 8.6. *Suppose f is analytic on the region G, $w \in G$, and γ is a positively oriented, simple, closed, smooth, G-contractible curve such that w is inside γ. Then*

$$f^{(k)}(w) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z - w)^{k+1}} \, dz \,.$$

Corollary 8.6 combined with our often-used Proposition 4.1(d) gives an inequality which is often called *Cauchy's Estimate*:

Corollary 8.7. *Suppose f is analytic in $\{z \in \mathbb{C} : |z - w| < R\}$ and $|f| \leq M$. Then*

$$\left| f^{(k)}(w) \right| \leq \frac{k!M}{R^k} \, .$$

Proof. Let γ be a circle centered at w with radius $r < R$. Then Corollary 8.6 applies, and we can estimate using Proposition 4.1(d):

$$\left| f^{(k)}(w) \right| = \left| \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z - w)^{k+1}} \, dz \right| \leq \frac{k!}{2\pi} \max_{z \in \gamma} \left| \frac{f(z)}{(z - w)^{k+1}} \right| \text{length}(\gamma) \leq \frac{k!}{2\pi} \frac{M}{r^{k+1}} 2\pi r = \frac{k!M}{r^k} \, .$$

The statement now follows since r can be chosen arbitrarily close to R. $\qquad\qquad\square$

8.2 Classification of Zeros and the Identity Principle

Basic algebra shows that if a polynomial $p(z)$ of positive degree d has a a zero at a (in other words, if $p(a) = 0$) then $p(z)$ has $z - a$ as a factor. That is, $p(z) = (z - a)q(z)$ where $q(z)$ is a polynomial of degree $d - 1$. We can then ask whether $q(z)$ itself has a zero at a and, if so, we can factor out another factor of $z - a$. continuing in this way we see that we can factor $p(z)$ as $p(z) = (z - a)^m g(z)$ where m is a positive integer, not bigger than d, and $g(z)$ is a polynomial which does not have a zero at a. The integer m is called the *multiplicity* of the zero a of $p(z)$.

Almost exactly the same thing happens for analytic functions:

Theorem 8.8 (Classification of Zeros). *Suppose f is an analytic function defined on an open set G and suppose f has a zero at a point a in G. Then there are exactly two possibilities:*

(a) *Either: f is identically zero on some open disk D centered at a (that is, $f(z) = 0$ for all z in D);*

(b) *or: there is a positive integer m and an analytic function g, defined on G, satisfying $f(z) = (z - a)^m g(z)$ for all z in G, with $g(a) \neq 0$*

The integer m in the second case is uniquely determined by f and a and is called the multiplicity *of the zero at a.*

Proof. We have a power series expansion for $f(z)$ in some disk $D_r(a)$ of radius r around a, so $f(z) = \sum_{k \geq 0} c_k (z - a)^k$, and $c_0 = f(0)$ is zero since a is a zero of f. There are now exactly two possibilities:

(a) Either $c_k = 0$ for all k;

(b) or there is some positive integer m so that $c_k = 0$ for all $k < m$ but $c_m \neq 0$.

The first case clearly gives us $f(z) = 0$ for all z in $D = D_r(a)$. So now consider the second case. Notice that

$$f(z) = c_m(z - a)^m + c_{m+1}(z - a)^{m+1} + \cdots = (z - a)^m \left(c_m + c_{m+1}(z - a) + \cdots \right)$$

$$= (z - a)^m \sum_{k \geq 0} c_{k+m}(z - a)^k.$$

Then we can define a function g on G by

$$g(z) = \begin{cases} \sum_{k \geq 0} c_{k+m}(z-a)^k & \text{if } |z-a| < r \\ \dfrac{f(z)}{(z-a)^m} & \text{if } z \in G \setminus \{a\} \end{cases}$$

According to our calculations above, the two definitions give the same value when both are applicable. The function g is analytic at a by the first definition; and g is analytic at other points of G by the second definition. Finally, $g(a) = c_m \neq 0$.

Clearly m is unique, since it is defined in terms of the power series expansion of f at a, which is unique. $\square$

The proof of this last theorem immediately yields the following.

Corollary 8.9. *Suppose f is an analytic function defined on an open set G and suppose f has a zero at $a \in G$. Then the multiplicity of a equals m if and only if we can write f as a power series of the form $f(z) = \sum_{k \geq m} c_k (z-a)^k$ with $c_m \neq 0$.*

To start using the intimate connection of analytic functions and power series, we apply Theorem 8.8 to obtain the following result, which is sometimes also called the *uniqueness theorem*.

Theorem 8.10 (Identity Principle). *Suppose f and g are analytic in the region G and $f(z_k) = g(z_k)$ at a sequence that converges to $a \in G$ with $z_k \neq a$ for all k. Then $f(z) = g(z)$ for all z in G.*

Proof. We start by defining $h = f - g$. Then h is analytic on G, $h(z_n) = 0$, and we will be finished if we can deduce that h is identically zero on G. Now notice the following: If b is in G then exactly one of the following occurs:

(a) Either there is an open disk D centered at b so that $h(z) = 0$ for all z in D;

(b) or there is an open disk D centered at b so that $h(z) \neq 0$ for all z in $D \setminus \{b\}$.

To see this, suppose that $h(b) \neq 0$. Then, by continuity, there is an open disk D centered at b so that $h(z) \neq 0$ for all $z \in D$, so b satisfies the second condition. If $h(b) = 0$ then, by the classification of zeros, either $h(z) = 0$ for all z in some open disk D centered at b, so b satisfies the first condition; or $h(z) = (z-b)^m \phi(z)$ for all z in G, where ϕ is analytic and $\phi(b) \neq 0$. Then, since ϕ is continuous, there is an open disk D centered at b so that $\phi(z) \neq 0$ for all z in D. Then $h(z) = (z-b)^m \phi(z) \neq 0$ for all z in D except $z = b$, so b satisfies the second condition.

Now define two sets $X, Y \subseteq G$, so that $b \in X$ if b satisfies the first condition above, and $b \in Y$ if b satisfies the second condition. If $b \in X$ and D is an open disk centered at b as in the first condition then it is clear that $D \subseteq X$. If $b \in Y$ and D is an open disk centered at b as in the second condition then $D \subseteq Y$, since if $z \in D \setminus \{b\}$ then $h(z) \neq 0$, and we saw that this means z satisfies the second condition.

Finally, we check that our original point a lies in X. To see this, suppose $a \in Y$, and let D be an open disk centered at a so that $h(z) \neq 0$ for all z in D except $z = b$. But, since the sequence z_k converges to a, there is some k so that z_k is in D, so $h(z_k) = 0$. Since $z_k \neq a$, this is a contradiction.

Now we finish the proof using the definition of connectedness. X and Y are disjoint open sets whose union is G, so one of them must be empty. Since a is in X, we must have $Y = \varnothing$ and $X = G$. But $X = G$ implies that every z in G satisfies the first condition above, so $h(z) = 0$. $\square$

Using the identity principle, we can prove yet another important property of analytic functions.

Theorem 8.11 (Maximum-Modulus Theorem). *Suppose f is analytic and not constant in the region G. Then $|f|$ does not attain a weak relative maximum in G.*

There are many reformulations of this theorem, such as: If G is a bounded region and f is analytic in the closure of G, then the maximum of $|f|$ is attained on the boundary of G.

Proof. Suppose there is a point a in G and an open disk D_0 centered at a so that $|f(z)| \leq |f(a)|$ for all z in D_0. If $f(a) = 0$ then $f(z) = 0$ for all z in D_0, so f is identically zero, by the identity principle. So we assume $f(a) \neq 0$. In this case we can define an analytic function $g(z) = f(z)/f(a)$, and we have the condition $|g(z)| \leq |g(a)| = 1$ for all z in D_0. Since $g(a) = 1$ we can find, using continuity, a smaller open disk D centered at a so that $g(z)$ has positive real part for all z in D. Thus the function $h = \text{Log} \circ g$ is defined and analytic on D, and we have $h(a) = \text{Log}(g(a)) = \text{Log}(1) = 0$ and $\text{Re}\, h(z) = \text{Re}\, \text{Log}(g(z)) = \ln(|g(z)|) \leq \ln(1) = 0$.

We now refer to Exercise 27, which shows that h must be identically zero in D. Hence $g(z) = e^{h(z)}$ must be equal to $e^0 = 1$ for all z in D, and so $f(z) = f(a)g(z)$ must have the constant value $f(a)$ for all z in D. Hence, by the identity principle, $f(z)$ has the constant value $f(a)$ for all z in G. □

Theorem 8.11 can be used to give a proof of the analogous theorem for harmonic functions, Theorem 6.5, in the process strengthening that theorem to cover weak maxima and weak minima.

Corollary 8.12. *If u is harmonic in the region G, then it does not have a weak relative maximum or minimum in G.*

Since the last corollary also covers minima of harmonic functions, we should not be too surprised to find the following result whose proof we leave for the exercises.

Corollary 8.13 (Minimum-Modulus Theorem). *Suppose f is analytic and not constant in the region G. Then $|f|$ does not attain a weak relative minimum at a in G unless $f(a) = 0$.*

8.3 Laurent Series

Theorem 8.5 gives a powerful way of describing analytic functions. It is, however, not as general as it could be. It is natural, for example, to think about representing $\exp\left(\frac{1}{z}\right)$ as

$$\exp\left(\frac{1}{z}\right) = \sum_{k \geq 0} \frac{1}{k!} \left(\frac{1}{z}\right)^k = \sum_{k \geq 0} \frac{1}{k!} z^{-k},$$

a "power series" with negative exponents. To make sense of expressions like the above, we introduce the concept of a *double series*

$$\sum_{k \in \mathbb{Z}} a_k = \sum_{k \geq 0} a_k + \sum_{k \geq 1} a_{-k}.$$

Here $a_k \in \mathbb{C}$ are terms indexed by the integers. A double series converges if both its defining series do. Absolute and uniform convergence are defined analogously. Equipped with this, we can now state the following central definition.

Definition 8.1. A *Laurent*[2] *series centered at* z_0 is a double series of the form $\sum_{k \in \mathbb{Z}} c_k (z - z_0)^k$.

Example 8.1. The series which started this section is the Laurent series of $\exp\left(\frac{1}{z}\right)$ centered at 0.

Example 8.2. Any power series is a Laurent series (with $c_k = 0$ for $k < 0$).

We should pause for a minute and ask for which z such a Laurent series can possibly converge. By definition

$$\sum_{k \in \mathbb{Z}} c_k (z - z_0)^k = \sum_{k \geq 0} c_k (z - z_0)^k + \sum_{k \geq 1} c_{-k} (z - z_0)^{-k}.$$

The first of the series on the right-hand side is a power series with some radius of convergence R_2, that is, it converges in $\{z \in \mathbb{C} : |z - z_0| < R_2\}$. The second we can view as a "power series in $\frac{1}{z - z_0}$," it will converge for $\frac{1}{z - z_0} < \frac{1}{R_1}$ for some R_1, that is, in $\{z \in \mathbb{C} : |z - z_0| > R_1\}$. For the convergence of our Laurent series, we need to combine those two notions, whence the Laurent series converges on the annulus $\{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$ (if $R_1 < R_2$). Even better, Theorem 7.14 implies that the convergence is uniform on a set of the form $\{z \in \mathbb{C} : r_1 \leq |z - z_0| \leq r_2\}$ for any $R_1 < r_1 < r_2 < R_2$. Theorem 8.1 says that the Laurent series represents a function which is analytic on $\{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$. The fact that we can conversely represent any function analytic in such an annulus by a Laurent series is the substance of the next theorem.

Theorem 8.14. *Suppose f is a function which is analytic in $A = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$. Then f can be represented in A as a Laurent series centered at z_0:*

$$f(z) = \sum_{k \in \mathbb{Z}} c_k (z - z_0)^k \qquad with \qquad c_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}} \, dw.$$

Here γ is any circle in A centered at z_0.

Remark. Naturally, by Cauchy's Theorem 4.4 we can replace the circle in the formula for the Laurent series by any closed, smooth path that is A-homotopic to the circle.

Proof. Let $g(z) = f(z + z_0)$; so g is a function analytic in $\{z \in \mathbb{C} : R_1 < |z| < R_2\}$. Fix $R_1 < r_1 < |z| < r_2 < R_2$, and let γ_1 and γ_2 be positively oriented circles centered at 0 with radii r_1 and r_2, respectively. By introducing an "extra piece" (see Figure 8.1), we can apply Cauchy's integral formula (Theorem 4.8) to the path $\gamma_2 - \gamma_1$:

$$g(z) = \frac{1}{2\pi i} \int_{\gamma_2 - \gamma_1} \frac{g(w)}{w - z} \, dw = \frac{1}{2\pi i} \int_{\gamma_2} \frac{g(w)}{w - z} \, dw - \frac{1}{2\pi i} \int_{\gamma_1} \frac{g(w)}{w - z} \, dw. \qquad (8.1)$$

For the integral over γ_2 we play exactly the same game as in Theorem 8.5. The factor $1/(w - z)$ in this integral can be expanded into a geometric series (note that $w \in \gamma_2$ and so $\left|\frac{z}{w}\right| < 1$)

$$\frac{1}{w - z} = \frac{1}{w} \frac{1}{1 - \frac{z}{w}} = \frac{1}{w} \sum_{k \geq 0} \left(\frac{z}{w}\right)^k,$$

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[2]For more information about Pierre Alphonse Laurent (1813–1854), see
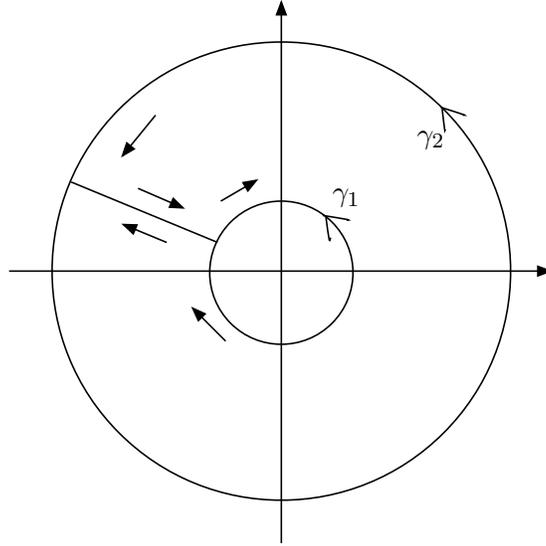`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Laurent_Pierre.html`.

Figure 8.1: Proof of Theorem 8.14.

which converges uniformly in the variable $w \in \gamma_2$ (by Lemma 7.13). Hence Proposition 7.9 applies:

$$\int_{\gamma_2} \frac{g(w)}{w-z}\, dw = \int_{\gamma_2} g(w) \frac{1}{w} \sum_{k \geq 0} \left(\frac{z}{w}\right)^k dw = \sum_{k \geq 0} \int_{\gamma_2} \frac{g(w)}{w^{k+1}}\, dw\, z^k.$$

The integral over γ_1 is computed in a similar fashion; now we expand the factor $1/(w-z)$ into the following geometric series (note that $w \in \gamma_1$ and so $\left|\frac{w}{z}\right| < 1$)

$$\frac{1}{w-z} = -\frac{1}{z} \frac{1}{1-\frac{w}{z}} = -\frac{1}{z} \sum_{k \geq 0} \left(\frac{w}{z}\right)^k,$$

which converges uniformly in the variable $w \in \gamma_1$ (by Lemma 7.13). Again Proposition 7.9 applies:

$$\int_{\gamma_1} \frac{g(w)}{w-z}\, dw = -\int_{\gamma_1} g(w) \frac{1}{z} \sum_{k \geq 0} \left(\frac{w}{z}\right)^k dw = -\sum_{k \geq 0} \int_{\gamma_1} g(w) w^k\, dw\, z^{-k-1} = -\sum_{k \leq -1} \int_{\gamma_1} \frac{g(w)}{w^{k+1}}\, dw\, z^k.$$

Putting everything back into (8.1) gives

$$g(z) = \frac{1}{2\pi i} \sum_{k \geq 0} \int_{\gamma_2} \frac{g(w)}{w^{k+1}}\, dw\, z^k + \sum_{k \leq -1} \int_{\gamma_1} \frac{g(w)}{w^{k+1}}\, dw\, z^k.$$

We can now change both integration paths to a circle γ centered at 0 with a radius between R_1 and R_2 (by Cauchy's Theorem 4.4), which finally gives

$$g(z) = \frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} \int_{\gamma} \frac{g(w)}{w^{k+1}}\, dw\, z^k.$$

The statement follows now with $f(z) = g(z - z_0)$ and an easy change of variables. $\qquad \square$

We finish this chapter with a consequence of the above theorem: because the coefficients of a Laurent series are given by integrals, we immediately obtain the following:

Corollary 8.15. *The coefficients of a Laurent series are unique.*

This result seems a bit artificial; what it says is simply the following: if we expand a function (that is analytic in some annulus) into a Laurent series, there is only one possible outcome.

Exercises

1. For each of the following series, determine where the series converges absolutely/uniformly:

   (a) $\sum_{k \geq 2} k(k-1) z^{k-2}$.

   (b) $\sum_{k \geq 0} \frac{1}{(2k+1)!} z^{2k+1}$.

   (c) $\sum_{k \geq 0} \left(\frac{1}{z-3} \right)^k$.

2. What functions are represented by the series in the previous exercise?

3. Find the power series centered at 1 for $\exp z$.

4. Prove Lemma 3.8 using the power series of $\exp z$ centered at 0.

5. By integrating a series for $\frac{1}{1+z^2}$ term by term, find a power series for $\arctan(z)$. What is its radius of convergence?

6. Find the terms through third order and the radius of convergence of the power series for each following functions, centered at z_0. *Do not find the general form for the coefficients.*

   (a) $f(z) = \frac{1}{1+z^2}$, $z_0 = 1$.

   (b) $f(z) = \frac{1}{e^z+1}$, $z_0 = 0$.

   (c) $f(z) = \sqrt{1+z}$, $z_0 = 0$ (use the principal branch).

   (d) $f(z) = e^{z^2}$, $z_0 = i$.

7. Prove the following generalization of Theorem 8.1: Suppose f_n are analytic on the region G and converge uniformly to f on G. Then f is analytic in G. (This result is called the *Weierstraß convergence theorem.*)

8. Use the previous exercise and Corollary 8.6 to prove the following: Suppose f_n are analytic on the region G and converge uniformly to f on G. Then for any $k \in \mathbb{N}$, the kth derivatives $f_n^{(k)}$ converge (pointwise) to $f^{(k)}$.

9. Prove the minimum-modulus theorem (Corollary 8.13).

10. Find the maximum and minimum of $|f(z)|$ on the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$, where $f(z) = z^2 - 2$.

11. Give another proof of the fundamental theorem of algebra (Theorem 5.4), using the minimum-modulus theorem (Corollary 8.13). (*Hint*: Use Lemma 5.3 to show that a polynomial does not achieve its minimum modulus on a large circle; then use the minimum-modulus theorem to deduce that the polynomial has a zero.)

12. Find a Laurent series for $\frac{1}{(z-1)(z+1)}$ centered at $z = 1$ and specify the region in which it converges.

13. Find a Laurent series for $\frac{1}{z(z-2)^2}$ centered at $z = 2$ and specify the region in which it converges.

14. Find a Laurent series for $\frac{z-2}{z+1}$ centered at $z = -1$ and specify the region in which it converges.

15. Find the first five terms in the Laurent series for $\frac{1}{\sin z}$ centered at $z = 0$.

16. Find the first 4 non-zero terms in the power series expansion of $\tan z$ centered at the origin. What is the radius of convergence?

17. (a) Find the power series representation for e^{az} centered at 0, where a is any constant.

(b) Show that $e^z \cos(z) = \frac{1}{2} \left(e^{(1+i)z} + e^{(1-i)z} \right)$.

(c) Find the power series expansion for $e^z \cos(z)$ centered at 0.

18. Show that $\frac{z-1}{z-2} = \sum_{k \geq 0} \frac{1}{(z-1)^k}$ for $|z - 1| > 1$.

19. Prove: If f is entire and $\mathrm{Im}(f)$ is constant on the unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$ then f is constant.

20. (a) Find the Laurent series for $\frac{\cos z}{z^2}$ centered at $z = 0$.

(b) Prove that

$$f(z) = \begin{cases} \frac{\cos z - 1}{z^2} & \text{if } z \neq 0, \\ -\frac{1}{2} & \text{if } z = 0 \end{cases}$$

is entire.

21. Suppose that $f(z)$ has a zero of multiplicity m at a. Explain why $\frac{1}{f(z)}$ has a pole of order m at a.

22. Suppose that $f(z_0) = 0$ and $f'(z_0) \neq 0$. Show that f has a zero of multiplicity 1 at z_0.

23. Find the multiplicities of the zeros:

(a) $f(z) = e^z - 1$, $z_0 = 2k\pi i$, where k is any integer.

(b) $f(z) = \sin(z) - \tan(z)$, $z_0 = 0$.

(c) $f(z) = \cos(z) - 1 + \frac{1}{2}\sin^2(z)$, $z_0 = 0$.

24. Find the zeros of the following, and determine their multiplicities:

(a) $(1 + z^2)^4$.

(b) $\sin^2 z$.

(c) $1 + e^z$.

(d) $z^3 \cos z$.

25. Find the poles of the following, and determine their orders:

(a) $(z^2 + 1)^{-3}(z - 1)^{-4}$.

(b) $z \cot(z)$.

(c) $z^{-5} \sin(z)$.

(d) $\frac{1}{1-e^z}$.

(e) $\frac{z}{1-e^z}$.

26. Suppose that $f(z)$ has exactly one zero, at a, inside the circle γ, and that it has multiplicity 1. Show that $a = \frac{1}{2\pi i} \int_\gamma \frac{zf'(z)}{f(z)} \, dz$.

27. Suppose f is analytic and not identically zero on an open disk D centered at a, and suppose $f(a) = 0$. Follow the following outline to show that $\operatorname{Re} f(z) > 0$ for some z in D.

(a) Why can you write $f(z) = (z - a)^m g(z)$ where $m > 0$, g is analytic, and $g(a) \neq 0$?

(b) Write $g(a)$ in polar form as $g(a) = c\,e^{i\alpha}$ and define $G(z) = e^{-i\alpha}g(z)$. Why is $\operatorname{Re} G(a) > 0$?

(c) Why is there a positive constant δ so that $\operatorname{Re} G(z) > 0$ for all z in the open disk $D_\delta(a)$?

(d) Write $z = a + re^{i\theta}$ for $0 < r < \delta$. Show that $f(z) = r^m e^{im\theta} e^{i\alpha} G(z)$.

(e) Find a value of θ so that $f(z)$ has positive real part.

28. Suppose $|c_n| \geq 2^n$ for all n. What can you say about the radius of convergence of $\sum_{k\geq 0} c_k z^k$?

29. Suppose the radius of convergence of $\sum_{k\geq 0} c_k z^k$ is R. What is the radius of convergence of each of the following?

(a) $\displaystyle\sum_{k\geq 0} k^2 c_k z^k$.

(b) $\displaystyle\sum_{k\geq 0} c_k z^{2k}$.

(c) $\displaystyle\sum_{k\geq 0} c_k z^{k+5}$.

(d) $\displaystyle\sum_{k\geq 0} 3^k c_k z^k$.

(e) $\displaystyle\sum_{k\geq 0} c_k^2 z^k$.

Chapter 9

Isolated Singularities and the Residue Theorem

$1/r^2$ has a nasty singularity at $r = 0$, but it did not bother Newton—the moon is far enough.
Edward Witten

9.1 Classification of Singularities

What is the difference between the functions $\frac{\sin z}{z}$, $\frac{1}{z^4}$, and $\exp\left(\frac{1}{z}\right)$? All of them are not defined at 0, but the singularities are of a very different nature. For complex functions there are three types of singularities, which are classified as follows.

Definition 9.1. If f is analytic in the punctured disk $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some $R > 0$ but not at $z = z_0$ then z_0 is an *isolated singularity* of f. The singularity z_0 is called

(a) *removable* if there is a function g analytic in $\{z \in \mathbb{C} : |z - z_0| < R\}$ such that $f = g$ in $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$,

(b) a *pole* if $\lim\limits_{z \to z_0} |f(z)| = \infty$,

(c) *essential* if z_0 is neither removable nor a pole.

Example 9.1. The function $\frac{\sin z}{z}$ has a removable singularity at 0, as for $z \neq 0$

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)!} z^{2k+1} = \sum_{k \geq 0} \frac{(-1)^k}{(2k+1)!} z^{2k}.$$

and the power series on the right-hand side represents an entire function (you may meditate on the fact why it has to be entire).

Example 9.2. The function $\frac{1}{z^4}$ has a pole at 0, as

$$\lim_{z \to 0} \left| \frac{1}{z^4} \right| = \infty.$$

Example 9.3. The function $\exp\left(\frac{1}{z}\right)$ does not have a removable singularity (consider, for example, $\lim_{x\to 0^+}\exp\left(\frac{1}{x}\right) = \infty$). On the other hand, $\exp\left(\frac{1}{z}\right)$ approaches 0 as z approaches 0 from the negative real axis. Hence $\lim_{z\to 0}\left|\exp\left(\frac{1}{z}\right)\right| \neq \infty$, that is, $\exp\left(\frac{1}{z}\right)$ has an essential singularity at 0.

To get a feel for the different types of singularities, we start with the following results.

Proposition 9.1. *Suppose z_0 is a isolated singularity of f. Then*

(a) *z_0 is removable if and only if $\lim_{z\to z_0} (z - z_0) f(z) = 0$;*

(b) *if z_0 is a pole then $\lim_{z\to z_0} (z - z_0)^{n+1} f(z) = 0$ for some positive integer n.*

Remark. The smallest possible n in (b) is the *order* of the pole. We will see in the proof that "near the pole z_0" we can write $f(z)$ as $\frac{h(z)}{(z-z_0)^n}$ for some function h which is analytic (and not zero) at z_0. This is very similar to the game we played with zeros in Chapter 8: f has a zero of order (or multiplicity) m at z_0 if we can write $f(z) = (z - z_0)^m h(z)$, where h is analytic and not zero at z_0. We will make use of the notions of zeros and poles of certain orders quite extensively in this chapter.

Proof. (a) Suppose z_0 is removable, and g is analytic on $D_R(z_0)$, the open disk with radius R centered at z_0 such that $f = g$ for $z \neq z_0$. Then we can make use of the fact that g is continuous at z_0:

$$\lim_{z\to z_0} (z - z_0) f(z) = \lim_{z\to z_0} (z - z_0) g(z) = g(z_0) \lim_{z\to z_0} (z - z_0) = 0.$$

Conversely, suppose that $\lim_{z\to z_0} (z - z_0) f(z) = 0$, and f is analytic on the punctured disk $\hat{D}_R(z_0) = D_R(z_0) \setminus \{z_0\}$. Then define

$$g(z) = \begin{cases} (z - z_0)^2 f(z) & \text{if } z \neq z_0, \\ 0 & \text{if } z = z_0. \end{cases}$$

Clearly g is analytic for $z \neq z_0$, and it is also differentiable at z_0, since we can calculate

$$g'(z_0) = \lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{(z - z_0)^2 f(z)}{z - z_0} = \lim_{z\to z_0} (z - z_0) f(z) = 0$$

So g is analytic in $D_R(z_0)$ with $g(z_0) = 0$ and $g'(z_0) = 0$, so it has a power series expansion $g(z) = \sum_{k\geq 0} c_k (z - z_0)^k$ with $c_0 = c_1 = 0$. Hence we can factor $(z - z_0)^2$ from the series, so

$$g(z) = (z - z_0)^2 \sum_{k\geq 0} c_{k+2}(z - z_0)^k = (z - z_0)^2 f(z).$$

Hence, for $z \neq z_0$, $f(z) = \sum_{k\geq 0} c_{k+2}(z - z_0)^k$, and this series defines an analytic function in $D_R(z_0)$.

(b) Suppose that z_0 is a pole of f. Then there is some $R > 0$ so that $|f(z)| > 1$ in the punctured disk $\hat{D}_R(z_0)$, and

$$\lim_{z\to z_0} \frac{1}{f(z)} = 0.$$

So, if we define $g(z)$ by

$$g(z) = \begin{cases} \frac{1}{f(z)} & \text{if } z \in \hat{D}_R(z_0), \\ 0 & \text{if } z = z_0, \end{cases}$$

then g is analytic in $D_R(z_0)$ (by part (a)). By the classification of zeros, $g(z) = (z - z_0)^n \phi(z)$ where ϕ is analytic in $D_R(z_0)$ and $\phi(z_0) \neq 0$. In fact, $\phi(z) \neq 0$ for all z in $D_R(z_0)$ since $g(z) \neq 0$ for $z \in \hat{D}_R(z_0)$. Hence $h = \frac{1}{\phi}$ is an analytic function in $D_R(z_0)$ and

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z - z_0)^n \phi(z)} = \frac{h(z)}{(z - z_0)^n} \, .$$

But then, since h is continuous at z_0,

$$\lim_{z \to z_0} (z - z_0)^{n+1} f(z) = \lim_{z \to z_0} (z - z_0) h(z) = h(z_0) \lim_{z \to z_0} (z - z_0) = 0 \, . \qquad \square$$

The reader might have noticed that the previous proposition did not include any result on essential singularities. Not only does the next theorem make up for this but it also nicely illustrates the strangeness of essential singularities. To appreciate the following result, we suggest meditating about its statement for a couple of minutes over a good cup of coffee.

Theorem 9.2 (Casorati[1]-Weierstraß). *If z_0 is an essential singularity of f and $D = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some $R > 0$, then any $w \in \mathbb{C}$ is arbitrarily close to a point in $f(D)$, that is, for any $w \in \mathbb{C}$ and any $\epsilon > 0$ there exists $z \in D$ such that $|w - f(z)| < \epsilon$.*

Remarks. 1. In the language of topology, the Casorati-Weierstraß theorem says that the image of any punctured disc centered at an essential singularity is *dense* in $\mathbb{C}$.

2. There is a much stronger theorem, which is beyond the scope of this book, and which implies the Casorati-Weierstraß theorem. It is due to Charles Emile Picard (1856–1941)[2] and says that the image of any punctured disc centered at an essential singularity misses at most one point of $\mathbb{C}$. (It is worth meditating about coming up with examples of functions which do not miss any point in $\mathbb{C}$ and functions which miss exactly one point. Try it!)

Proof. Suppose (by way of contradiction) that there is a $w \in \mathbb{C}$ and an $\epsilon > 0$ such that for all z in the punctured disc D (centered at z_0)

$$|w - f(z)| \geq \epsilon \, .$$

Then the function $g(z) = \frac{1}{(f(z) - w)}$ stays bounded as $z \to z_0$, and so

$$\lim_{z \to z_0} (z - z_0) g(z) = \lim_{z \to z_0} \frac{z - z_0}{f(z) - w} = 0 \, .$$

(The previous proposition tells us that g has a removable singularity at z_0.) Hence

$$\lim_{z \to z_0} \left| \frac{f(z) - w}{z - z_0} \right| = \infty \, .$$

But this implies that f has a pole or a removable singularity at z_0, which is a contradiction. $\qquad \square$

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[1] For more information about Felice Casorati (1835–1890), see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Casorati.html`.
[2] For more information about Picard, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Picard_Emile.html`.

Definition 9.1 is not always handy. The following classifies singularities according to their Laurent series.

Proposition 9.3. *Suppose z_0 is an isolated singularity of f with Laurent series*

$$f(z) = \sum_{k \in \mathbb{Z}} c_k (z - z_0)^k$$

(valid in $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ for some $R > 0$). Then

(a) z_0 *is removable if and only if there are no negative exponents (that is, the Laurent series is a power series);*

(b) z_0 *is a pole if and only if there are finitely many negative exponents;*

(c) z_0 *is essential if and only if there are infinitely many negative exponents.*

Proof. (a) Suppose z_0 is removable, and g is analytic on $\{z \in \mathbb{C} : |z - z_0| < R\}$ such that $f = g$ in $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Then the Laurent series of g in this region is a power series, and by Corollary 8.15 (uniqueness theorem for Laurent series) it has to coincide with the Laurent series of f. Conversely, if the Laurent series of f at z_0 has only nonnegative powers, we can use it to define a function which is analytic at z_0.

(b) Suppose z_0 is a pole of order n. Then by Proposition 9.1, the function $(z - z_0)^n f(z)$ has a removable singularity at z_0. By part (a), we can hence expand

$$(z - z_0)^n f(z) = \sum_{k \geq 0} c_k (z - z_0)^k,$$

that is,

$$f(z) = \sum_{k \geq 0} c_k (z - z_0)^{k-n} = \sum_{k \geq -n} c_k (z - z_0)^k.$$

Conversely, suppose that

$$f(z) = \sum_{k \geq -n} c_k (z - z_0)^k = (z - z_0)^{-n} \sum_{k \geq -n} c_k (z - z_0)^{k+n} = (z - z_0)^{-n} \sum_{k \geq 0} c_{k-n} (z - z_0)^k,$$

where $c_{-n} \neq 0$. Define

$$g(z) = \sum_{k \geq 0} c_{k-n} (z - z_0)^k.$$

Then since $g(z_0) = c_{-n} \neq 0$,

$$\lim_{z \to z_0} |f(z)| = \lim_{z \to z_0} \left| \frac{g(z)}{(z - z_0)^n} \right| = \infty.$$

(c) This follows by definition: an essential singularity is neither removable nor a pole. □

9.2   Residues

Suppose z_0 is an isolated singularity of f, γ is a positively oriented, simple, closed, smooth path around z_0, which lies in the domain of the Laurent series of f at z_0. Then—essentially by Proposition 7.9—we can integrate term by term:

$$\int_\gamma f = \int_\gamma \sum_{k \in \mathbb{Z}} c_k (z - z_0)^k \, dz = \sum_{k \in \mathbb{Z}} c_k \int_\gamma (z - z_0)^k \, dz \, .$$

The integrals inside the summation are easy: for nonnegative powers k the integral $\int_\gamma (z - z_0)^k$ is 0 (because $(z - z_0)^k$ is entire), and the same holds for $k \leq -2$ (because $(z - z_0)^k$ has a primitive on $\mathbb{C} \setminus \{z_0\}$). Finally, for $k = -1$ we can use Exercise 8 of Chapter 4. Because all the other terms give a zero integral, c_{-1} is the only term of the series which survives:

$$\int_\gamma f = \sum_{k \in \mathbb{Z}} c_k \int_\gamma (z - z_0)^k \, dz = 2\pi i \, c_{-1} \, .$$

(One might also notice that Theorem 8.14 gives the same identity.) Reason enough to give the c_{-1}-coefficient of a Laurent series a special name.

Definition 9.2. Suppose z_0 is an isolated singularity of f with Laurent series $\sum_{k \in \mathbb{Z}} c_k (z - z_0)^k$. Then c_{-1} is the *residue of f at z_0*, denoted by $\operatorname{Res}_{z=z_0}(f(z))$ or $\operatorname{Res}(f(z), z = z_0)$.

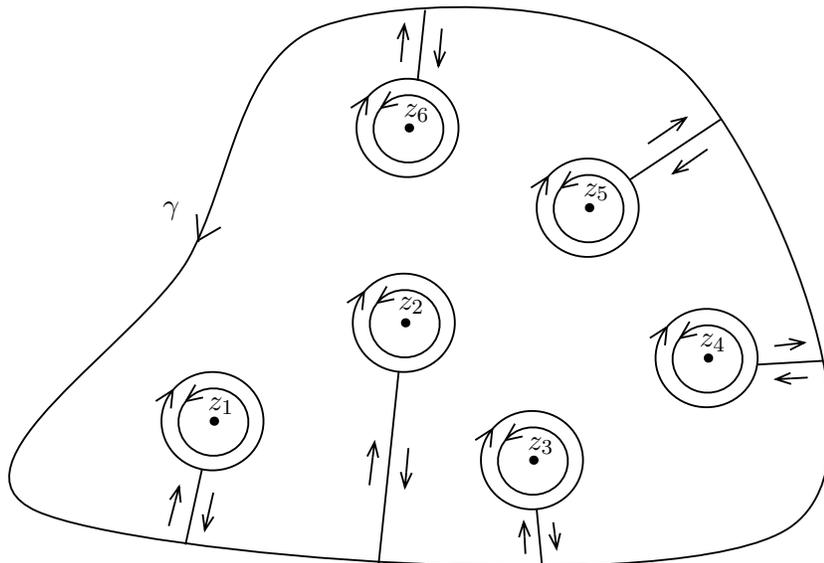The following theorem generalizes the discussion at the beginning of this section.

Figure 9.1: Proof of Theorem 9.4.

Theorem 9.4 (Residue Theorem)**.** *Suppose f is analytic in the region G, except for isolated singularities, and γ is a positively oriented, simple, closed, smooth, G-contractible curve. Then*

$$\int_\gamma f = 2\pi i \sum_k \operatorname{Res}_{z=z_k}(f(z)),$$

where the sum is taken over all singularities z_k inside γ.

Proof. Draw two circles around each isolated singularity inside γ, one with positive, and one with negative orientation, as pictured in Figure 9.1. Each of these pairs cancel each other when we integrate over them. Now connect the circles with negative orientation with γ. This gives a curve which is contractible in the region of analyticity of f. But this means that we can replace γ by the positively oriented circles; now all we need to do is described at the beginning of this section. $\qquad\square$

Computing integrals is as easy (or hard!) as computing residues. The following two lemmas start the range of tricks one can use when computing residues.

Lemma 9.5. *Suppose f and g are analytic in a region containing z_0, which is a simple zero of g, and $f(z_0) \neq 0$. Then $\frac{f}{g}$ has a simple pole at z_0 and*

$$\operatorname{Res}_{z=z_0}\left(\frac{f(z)}{g(z)}\right) = \frac{f(z_0)}{g'(z_0)}.$$

Proof. The functions f and g have power series centered at z_0; the one for g has by assumption no constant term:

$$g(z) = \sum_{k \geq 1} c_k (z - z_0)^k = (z - z_0) \sum_{k \geq 1} c_k (z - z_0)^{k-1}.$$

The series on the right represents an analytic function, call it h; note that $h(z_0) = c_1 \neq 0$. Hence

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z - z_0)h(z)},$$

and the function $\frac{f}{h}$ is analytic at z_0. Even more, the residue of $\frac{f}{g}$ equals the constant term of the power series of $\frac{f}{h}$ (that's how we get the (-1)st term of $\frac{f}{g}$). But this constant term is computed, as always, by $\frac{f(z_0)}{h(z_0)}$. But $h(z_0)$, in turn, is the constant term of h or the second term of g, which by Taylor's formula (Corollary 8.3) equals $g'(z_0)$. $\qquad\square$

Lemma 9.6. *Suppose z_0 is a pole of f of order n. Then*

$$\operatorname{Res}_{z=z_0}(f(z)) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}\left((z - z_0)^n f(z)\right).$$

Proof. We know by Proposition 9.3 that the Laurent series at z_0 looks like

$$f(z) = \sum_{k \geq -n} c_k (z - z_0)^k.$$

But then

$$(z - z_0)^n f(z) = \sum_{k \geq -n} c_k (z - z_0)^{k+n}$$

represents a power series, and we can use Taylor's formula (Corollary 8.3) to compute c_{-1}. $\qquad\square$

9.3 Argument Principle and Rouché's Theorem

There are many situations where we want to restrict ourselves to functions which are analytic in some region except possibly for poles. Such functions are called *meromorphic*. In this section, we will study these functions, especially with respect to their zeros and poles, which—as the reader might have guessed already—can be thought of as siblings.

Suppose we have a differentiable function f. Differentiating $\mathcal{L}\text{og}\, f$ (where $\mathcal{L}\text{og}$ is a branch of the logarithm) gives $\frac{f'}{f}$, which is one good reason why this quotient is called the *logarithmic derivative* of f. It has some remarkable properties, one of which we would like to discuss here.

Let's say we have two functions f and g analytic in some region. Then the logarithmic derivative of their product behaves very nicely:

$$\frac{(fg)'}{fg} = \frac{f'g + fg'}{fg} = \frac{f'}{f} + \frac{g'}{g}\,.$$

We can apply this fact to the following situation: Suppose that f is analytic on the region G, and f has the (finitely many) zeros $z_1, \ldots, z_j$ of order $n_1, \ldots, n_j$, respectively. Then we can express f as

$$f(z) = (z - z_1)^{n_1} \cdots (z - z_j)^{n_j} g(z)\,,$$

where g is also analytic in G and never zero. Let's compute the logarithmic derivative of f and play the same remarkable cancellation game as above:

$$\frac{f'(z)}{f(z)} = \frac{n_1(z - z_1)^{n_1-1}(z - z_2)^{n_2} \cdots (z - z_j)^{n_j} g(z) + \cdots + (z - z_1)^{n_1} \cdots (z - z_j)^{n_j} g'(z)}{(z - z_1)^{n_1} \cdots (z - z_j)^{n_j} g(z)}$$

$$= \frac{n_1}{z - z_1} + \frac{n_2}{z - z_2} + \ldots \frac{n_j}{z - z_j} + \frac{g'(z)}{g(z)}\,.$$

Something similar happens to the poles of f. We invite the reader to prove that if $p_1, \ldots, p_k$ are all the poles of f in G with order $m_1, \ldots, m_k$, respectively, then the logarithmic derivative of f can be expressed as

$$\frac{f'(z)}{f(z)} = -\frac{m_1}{z - p_1} - \frac{m_2}{z - p_2} - \cdots - \frac{m_k}{z - p_k} + \frac{g'(z)}{g(z)}\,, \tag{9.1}$$

where g is a function without poles in G. Naturally, we can combine the expressions we got for zeros and poles, which is the starting point of the following theorem.

Theorem 9.7 (Argument Principle). *Suppose f is meromorphic in the region G and γ is a positively oriented, simple, closed, smooth, G-contractible curve, which does not pass through any zero or pole of f. Denote by $Z(f, \gamma)$ the number of zeros of f inside γ—counted according to multiplicity— and by $P(f, \gamma)$ the number of poles of f inside γ, again counted according to multiplicity. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} = Z(f, \gamma) - P(f, \gamma)\,.$$

Proof. Suppose the zeros of f inside γ are $z_1, \ldots, z_j$ of order $n_1, \ldots, n_j$, respectively, and the poles inside γ are $p_1, \ldots, p_k$ with order $m_1, \ldots, m_k$, respectively. (You may meditate about the fact why there can only be finitely many zeros and poles inside γ.) In fact, we may shrink G, if necessary, so

that these are the only zeros and poles in G. Our discussion before the statement of the theorem yielded that the logarithmic derivative of f can be expressed as

$$\frac{f'(z)}{f(z)} = \frac{n_1}{z - z_1} + \cdots + \frac{n_j}{z - z_j} - \frac{m_1}{z - p_1} - \cdots - \frac{m_k}{z - p_k} + \frac{g'(z)}{g(z)},$$

where g is a function which is analytic in G (in particular, without poles) and never zero. Thanks to Exercise 8 of Chapter 4, the integral is easy:

$$\int_\gamma \frac{f'}{f} = n_1 \int_\gamma \frac{dz}{z - z_1} + \cdots + n_j \int_\gamma \frac{dz}{z - z_j} - m_1 \int_\gamma \frac{dz}{z - p_1} - \cdots - m_k \int_\gamma \frac{dz}{z - p_k} + \int_\gamma \frac{g'}{g}$$

$$= 2\pi i \left(n_1 + \cdots + n_j - m_1 - \cdots - m_k \right) + \int_\gamma \frac{g'}{g}.$$

Finally, $\frac{g'}{g}$ is analytic in G (recall that g is never zero in G), so that Corollary 4.5 (to Cauchy's Theorem 4.4) gives that

$$\int_\gamma \frac{g'}{g} = 0. \qquad \square$$

As a nice application of the argument principle, we present a famous theorem due to Eugene Rouché (1832–1910)[3].

Theorem 9.8 (Rouché's Theorem). *Suppose f and g are analytic in a region G, and γ is a positively oriented, simple, closed, smooth, G-contractible curve such that for all $z \in \gamma$, $|f(z)| > |g(z)|$. Then*

$$Z(f + g, \gamma) = Z(f, \gamma).$$

This theorem is of surprising practicality. It allows us to locate the zeros of a function fairly precisely. As an illustration, we prove:

Example 9.4. All the roots of the polynomial $p(z) = z^5 + z^4 + z^3 + z^2 + z + 1$ have absolute value less than two.[4] To see this, let $f(z) = z^5$ and $g(z) = z^4 + z^3 + z^2 + z + 1$, and let γ denote the circle centered at the origin with radius 2. Then for $z \in \gamma$

$$|g(z)| \le |z|^4 + |z|^3 + |z|^2 + |z| + 1 = 16 + 8 + 4 + 2 + 1 = 31 < 32 = |z|^5 = |f(z)|.$$

So g and f satisfy the condition of the Theorem 9.8. But f has just a root of order 5 at the origin, whence

$$Z(p, \gamma) = Z(f + g, \gamma) = Z(f, \gamma) = 5.$$

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[3]For more information about Rouché, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Rouche.html`.

[4]The fundamental theorem of algebra (Theorem 5.4) asserts that p has five roots in $\mathbb{C}$. What's special about the statement of Example 9.4 is that they all have absolute value < 2. Note also that there is no general formula for computing roots of a polynomial of degree 5. (Although for this p it's not hard to find one root—and therefore all of them.)

Proof of Theorem 9.8. By our analysis in the beginning of this section and by the argument principle (Theorem 9.7)

$$Z(f+g,\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{f+g} = \frac{1}{2\pi i}\int_\gamma \frac{\left(f\left(1+\frac{g}{f}\right)\right)'}{f\left(1+\frac{g}{f}\right)} = \frac{1}{2\pi i}\int_\gamma \left(\frac{f'}{f} + \frac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}}\right)$$

$$= Z(f,\gamma) + \frac{1}{2\pi i}\int_\gamma \frac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}}\,.$$

We are assuming that $\left|\frac{g}{f}\right| < 1$ on γ, which means that the function $1+\frac{g}{f}$ evaluated on γ stays away from the nonpositive real axis. But then $\mathrm{Log}\left(1+\frac{g}{f}\right)$ is a well defined analytic function on γ.

Its derivative is $\dfrac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}}$, which implies by Corollary 4.3 that

$$\frac{1}{2\pi i}\int_\gamma \frac{\left(1+\frac{g}{f}\right)'}{1+\frac{g}{f}} = 0\,. \qquad \square$$

Exercises

1. Prove (9.1).

2. Show that if f has an essential singularity at z_0 then $\frac{1}{f}$ also has an essential singularity at z_0.

3. Suppose f is a non-constant entire function. Prove that any complex number is arbitrarily close to a number in $f(\mathbb{C})$. (*Hint*: If f is not a polynomial, use Theorem 9.2 for $f\left(\frac{1}{z}\right)$.)

4. Suppose f is meromorphic in the region G, g is analytic in G, and γ is a positively oriented, simple, closed, G-contractible curve, which does not pass through any zero or pole of f. Denote the zeros and poles of f inside γ by $z_1,\ldots,z_j$ and $p_1,\ldots,p_k$, respectively, counted according to multiplicity. Prove that

$$\frac{1}{2\pi i}\int_\gamma g\,\frac{f'}{f} = \sum_{m=1}^{j} g(z_m) - \sum_{n=1}^{k} g(p_n)\,.$$

5. Find the number of zeros of

   (a) $3\exp z - z$ in $\{z \in \mathbb{C} : |z| \leq 1\}$;
   (b) $\frac{1}{3}\exp z - z$ in $\{z \in \mathbb{C} : |z| \leq 1\}$;
   (c) $z^4 - 5z + 1$ in $\{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$.

6. Give another proof of the fundamental theorem of algebra (Theorem 5.4), using Rouché's Theorem 9.8. (*Hint*: If $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + 1$, let $f(z) = a_n z^n$ and $g(z) = a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + 1$, and choose as γ a circle which is large enough to make the condition of Rouché's theorem work. You might want to first apply Lemma 5.3 to $g(z)$.)

7. (a) Find a Laurent series for $\frac{1}{(z^2-4)(z-2)}$ centered at $z = 2$ and specify the region in which it converges.

   (b) Compute $\int_\gamma \frac{dz}{(z^2-4)(z-2)}$, where γ is the positively oriented circle centered at 2 of radius 1.

8. Evaluate the following integrals for $\gamma(t) = 3\,e^{it}$, $0 \le t \le 2\pi$.

   (a) $\displaystyle\int_\gamma \cot z\, dz$

   (b) $\displaystyle\int_\gamma z^3 \cos\left(\frac{3}{z}\right) dz$

   (c) $\displaystyle\int_\gamma \frac{dz}{(z+4)(z^2+1)}$

   (d) $\displaystyle\int_\gamma z^2 \exp\left(\frac{1}{z}\right) dz$

   (e) $\displaystyle\int_\gamma \frac{\exp z}{\sinh z}\, dz$

   (f) $\displaystyle\int_\gamma \frac{i^{z+4}}{(z^2+16)^2}\, dz$

9. (a) Find the power series of $\exp z$ centered at $z = -1$.

   (b) Find $\int_\gamma \frac{\exp z}{(z+1)^{34}}\, dz$, where γ is the circle $|z+2| = 2$, positively oriented.

10. Suppose f has a simple pole (i.e., a pole of order 1) at z_0 and g is analytic at z_0. Prove that
$$\mathrm{Res}_{z=z_0}\big(f(z)g(z)\big) = g(z_0) \cdot \mathrm{Res}_{z=z_0}\big(f(z)\big).$$

11. Find the residue of each function at 0:

    (a) $z^{-3}\cos(z)$.

    (b) $\csc(z)$.

    (c) $\dfrac{z^2 + 4z + 5}{z^2 + z}$.

    (d) $e^{1-\frac{1}{z}}$.

    (e) $\dfrac{e^{4z} - 1}{\sin^2 z}$.

12. Use residues to evaluate the following:

    (a) $\displaystyle\int_\gamma \frac{dz}{z^4 + 4}$, where γ is the circle $|z + 1 - i| = 1$.

    (b) $\displaystyle\int_\gamma \frac{dz}{z(z^2 + z - 2)}$, where γ is the circle $|z - i| = 2$.

    (c) $\displaystyle\int_\gamma \frac{e^z\, dz}{z^3 + z}$, where γ is the circle $|z| = 2$.

(d) $\int_\gamma \dfrac{dz}{z^2 \sin z}$, where γ is the circle $|z| = 1$.

13. Suppose f has an isolated singularity at z_0.

    (a) Show that f' also has an isolated singularity at z_0.

    (b) Find $\operatorname{Res}_{z=z_0}(f')$.

14. Given $R > 0$, let γ_R be the half circle defined by $\gamma_R(t) = Re^{it}$, $0 \le t \le \pi$, and Γ_R be the closed curve composed of γ_R and the line segment $[-R, R]$.

    (a) Compute $\int_{\Gamma_R} \frac{dz}{(1+z^2)^2}$.

    (b) Prove that $\lim_{R\to\infty} \int_{\gamma_R} \frac{dz}{(1+z^2)^2} = 0$.

    (c) Combine (a) and (b) to evaluate the real integral $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$.

15. Suppose f is entire, and $a, b \in \mathbb{C}$ with $|a|, |b| < R$. Let γ be the circle centered at 0 with radius R. Evaluate

$$\int_\gamma \frac{f(z)}{(z-a)(z-b)}\, dz\,,$$

and use this to give an alternate proof of Liouville's Theorem 5.5. (*Hint*: Show that if f is bounded then the above integral goes to zero as R increases.)

Chapter 10

Discreet Applications of the Residue Theorem

> *All means (even continuous) sanctify the discrete end.*
> Doron Zeilberger

On the surface, this chapter is just a collection of exercises. They are more involved than any of the ones we've given so far at the end of each chapter, which is one reason why we lead the reader through each of the following ones step by step. On the other hand, these sections should really be thought of as a continuation of the lecture notes, just in a different format. All of the following 'problems' are of a *discrete* mathematical nature, and we invite the reader to solve them using *continuous* methods—namely, complex integration. It might be that there is no other result which so intimately combines discrete and continuous mathematics as does the Residue Theorem 9.4.

10.1 Infinite Sums

In this exercise, we evaluate—as an example—the sums $\sum_{k \geq 1} \frac{1}{k^2}$ and $\sum_{k \geq 1} \frac{(-1)^k}{k^2}$. We hope the idea how to compute such sums in general will become clear.

1. Consider the function $f(z) = \dfrac{\pi \cot(\pi z)}{z^2}$. Compute the residues at all the singularities of f.

2. Let N be a positive integer and γ_N be the rectangular curve from $N+1/2-iN$ to $N+1/2+iN$ to $-N-1/2+iN$ to $-N-1/2-iN$ back to $N+1/2-iN$.

   (a) Show that for all $z \in \gamma_N$, $|\cot(\pi z)| < 2$. (Use Exercise 21 in Chapter 3.)

   (b) Show that $\lim_{N \to \infty} \int_{\gamma_N} f = 0$.

3. Use the Residue Theorem 9.4 to arrive at an identity for $\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{k^2}$.

4. Evaluate $\sum_{k \geq 1} \frac{1}{k^2}$.

5. Repeat the exercise with the function $f(z) = \frac{\pi}{z^2 \sin(\pi z)}$ to arrive at an evaluation of

$$\sum_{k \geq 1} \frac{(-1)^k}{k^2}.$$

(*Hint*: To bound this function, you may use the fact that $1/\sin^2 z = 1 + \cot^2 z$.)

6. Evaluate $\sum_{k\geq 1} \frac{1}{k^4}$ and $\sum_{k\geq 1} \frac{(-1)^k}{k^4}$.

10.2   Binomial Coefficients

The binomial coefficient $\binom{n}{k}$ is a natural candidate for being explored analytically, as the binomial theorem[1] tells us that $\binom{n}{k}$ is the coefficient of z^k in $(1+z)^n$. As an example, we outline a proof of the identity (for $-1/4 < x < 1/4$)

$$\sum_{k\geq 0} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}.$$

1. Convince yourself that

$$\binom{2k}{k} = \frac{1}{2\pi i} \int_\gamma \frac{(1+w)^{2k}}{w^{k+1}} \, dw \, ,$$

   where γ is any simple closed curve such that 0 is inside γ.

2. Suppose $|x| < 1/4$. Find a simple closed curve γ surrounding the origin such that

$$\sum_{k\geq 0} \left(\frac{(1+w)^2}{w} x \right)^k$$

   converges uniformly on γ (as a function in w). Evaluate this sum.

3. Convince yourself that

$$\sum_{k\geq 0} \binom{2k}{k} x^k = \frac{1}{2\pi i} \sum_{k\geq 0} \int_\gamma \frac{(1+w)^{2k}}{w^k} x^k \, \frac{dw}{w} \, ,$$

   use 2. to interchange summation and integral, and use the Residue Theorem 9.4 to evaluate the integral.

10.3   Fibonacci Numbers

The *Fibonacci*[2] *numbers* are a sequence of integers defined recursively as:

$$\begin{aligned}
f_0 &= 1, \\
f_1 &= 1, \\
f_n &= f_{n-1} + f_{n-2} \qquad \text{for } n \geq 2.
\end{aligned}$$

Let $F(z) = \sum_{k\geq 0} f_n z^n$.

---

[1] The binomial theorem says that for $x, y \in \mathbb{C}$ and $n \in \mathbb{N}$, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

[2] For more information about Leonardo Pisano Fibonacci (1170–1250), see
http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Fibonacci.html.

1. Show that F has a positive radius of convergence.

2. Show that the recurrence relation among the f_n implies that $F(z) = \frac{1}{1-z-z^2}$. (*Hint*: Write down the power series of $zF(z)$ and $z^2F(z)$ and rearrange both so that you can easily add.)

3. Verify that $\text{Res}_{z=0}\left(\frac{1}{z^{n+1}(1-z-z^2)}\right) = f_n$.

4. Use the Residue Theorem 9.4 to derive an identity for f_n. (*Hint*: Integrate $\frac{1}{z^{n+1}(1-z-z^2)}$ around a circle with center 0 and radius R, and show that this integral vanishes as $R \to \infty$.)

5. Generalize to other recurrence relations.

10.4 The 'Coin-Exchange Problem'

In this exercise, we will solve and extend a classical problem of Ferdinand Georg Frobenius (1849–1917)[3]. Suppose a and b are relatively prime[4] positive integers, and t is a positive integer. Consider the function

$$f(z) = \frac{1}{(1-z^a)(1-z^b)z^{t+1}}\,.$$

1. Compute the residues at all non-zero poles of f.

2. Verify that $\text{Res}_{z=0}(f) = N(t)$, where

$$N(t) = \#\left\{(m,n) \in \mathbb{Z} : m,n \geq 0,\ ma+nb = t\right\}.$$

3. Use the Residue Theorem 9.4 to derive an identity for $N(t)$. (*Hint*: Integrate f around a circle with center 0 and radius R, and show that this integral vanishes as $R \to \infty$.)

4. Use the following three steps to simplify this identity to

$$N(t) = \frac{t}{ab} - \left\{\frac{b^{-1}t}{a}\right\} - \left\{\frac{a^{-1}t}{b}\right\} + 1\,.$$

Here, $\{x\}$ denotes the fractional part[5] of x, and $a^{-1}a \equiv 1 \pmod{b}$[6], and $b^{-1}b \equiv 1 \pmod{a}$.

(a) Verify that for $b = 1$,

$$N(t) = \#\left\{(m,n) \in \mathbb{Z} : m,n \geq 0,\ ma+n = t\right\} = \#\left\{m \in \mathbb{Z} : m \geq 0,\ ma \leq t\right\}$$

$$= \#\left(\left[0, \frac{t}{a}\right] \cap \mathbb{Z}\right) = \frac{t}{a} - \left\{\frac{t}{a}\right\} + 1\,.$$

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[3]For more information about Frobenius, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Frobenius.html`.

[4]this means that the integers don't have any common factor

[5]The fractional part of a real number x is, loosely speaking, the "part after the decimal point." More thoroughly, the *greatest integer function* of x, denoted by $\lfloor x \rfloor$, is the greatest integer not exceeding x. The fractional part is then $\{x\} = x - \lfloor x \rfloor$.

[6]This means that a^{-1} is an integer such that $a^{-1}a = 1 + kb$ for some $k \in \mathbb{Z}$.

(b) Use this together with the identity found in 3. to obtain

$$\frac{1}{a}\sum_{k=1}^{a-1}\frac{1}{(1-e^{2\pi ik/a})e^{2\pi ikt/a}} = -\left\{\frac{t}{a}\right\} + \frac{1}{2} - \frac{1}{2a}\,.$$

(c) Verify that

$$\sum_{k=1}^{a-1}\frac{1}{(1-e^{2\pi ikb/a})e^{2\pi ikt/a}} = \sum_{k=1}^{a-1}\frac{1}{(1-e^{2\pi ik/a})e^{2\pi ikb^{-1}t/a}}\,.$$

5. Prove that $N(ab - a - b) = 0$, and $N(t) > 0$ for all $t > ab - a - b$.

6. More generally, prove that, if k is a nonnegative integer, $N\left((k+1)ab - a - b\right) = k$, and $N(t) > k$ for all $t > (k+1)ab - a - b$.

Historical remark. Given relatively prime positive integers $a_1, \ldots, a_n$, let's call an integer t *representable* if there exist nonnegative integers $m_1, \ldots, m_n$ such that

$$t = \sum_{j=1}^{n} m_j a_j\,.$$

In the late 19th century, Frobenius raised the problem of finding the largest integer which is not representable. We call this largest integer the *Frobenius number* $g(a_1, \ldots, a_n)$. It is well known (probably at least since the 1880's, when James Joseph Sylvester (1814–1897)[7] studied the Frobenius problem) that $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$. We verified this result in 5. For $n > 2$, there is no known closed formula for $g(a_1, \ldots, a_n)$. The formula in 4. is due to Popoviciu. The notion of an integer being representable k times and the respective formula obtained in 6. can only be found in the most recent literature.

10.5 Dedekind sums

This exercise outlines yet another nontraditional application of the Residue Theorem 9.4. Given two positive, relatively prime integers a and b, let

$$f(z) = \cot(\pi az)\cot(\pi bz)\cot(\pi z)\,.$$

1. Choose an $\epsilon > 0$ such that the rectangular path γ_R from $1 - \epsilon - iR$ to $1 - \epsilon + iR$ to $-\epsilon + iR$ to $-\epsilon - iR$ back to $1 - \epsilon - iR$ does not pass through any of the poles of f.

(a) Compute the residues for the poles of f inside γ_R.
    Hint: use the periodicity of the cotangent and the fact that

$$\cot z = \frac{1}{z} - \frac{1}{3}z + \text{ higher-order terms }\,.$$

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[7]For more information about Sylvester, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Sylvester.html`.

(b) Prove that $\lim_{R\to\infty} \int_{\gamma_R} f = -2i$ and deduce that for any $R > 0$

$$\int_{\gamma_R} f = -2i \,.$$

2. Define

$$s(a,b) = \frac{1}{4b} \sum_{k=1}^{b-1} \cot\left(\frac{\pi k a}{b}\right) \cot\left(\frac{\pi k}{b}\right) \,. \tag{10.1}$$

Use the Residue Theorem 9.4 to show that

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) \,. \tag{10.2}$$

3. Can you generalize (10.1) and (10.2)?

Historical remark. The sum (10.1) is called a *Dedekind*[8] *sum.* It first appeared in the study of the *Dedekind η-function*

$$\eta(z) = \exp\left(\tfrac{\pi i z}{12}\right) \prod_{k\geq 1} (1 - \exp(2\pi i k z))$$

in the 1870's and has since intrigued mathematicians from such different areas as topology, number theory, and discrete geometry. The *reciprocity law* (10.2) is the most important and famous identity of the Dedekind sum. The proof that is outlined here is due to Hans Rademacher (1892–1969)[9].

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[8]For more information about Julius Wilhelm Richard Dedekind (1831–1916), see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Dedekind.html`.
[9]For more information about Rademacher, see
`http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Rademacher.html`.

Solutions to Selected Exercises

Chapter 1

1. (b) $\frac{19}{25} - \frac{8}{25}i$
(c) 1
(d) 1 if $n = 4k$, $k \in \mathbb{Z}$; i if $n = 1 + 4k$, $k \in \mathbb{Z}$; -1 if $n = 2 + 4k$, $k \in \mathbb{Z}$; $-i$ if $n = 3 + 4k$, $k \in \mathbb{Z}$.
2. (a) $\sqrt{5}$, $-2 - i$
(b) $5\sqrt{5}$, $5 - 10i$
(c) $\sqrt{\frac{10}{11}}$, $\frac{3}{11}(\sqrt{2} - 1) + \frac{i}{11}(\sqrt{2} + 9)$
(d) 8, $8i$
3. (a) $2e^{i\frac{\pi}{2}}$
(b) $\sqrt{2}e^{i\frac{\pi}{4}}$
(c) $2\sqrt{3}e^{i\frac{5\pi}{6}}$
4. (a) $-1 + i$
(b) $34i$
(c) -1
5. (a) $z = e^{i\frac{\pi}{3}k}$, $k = 0, 1, \ldots, 5$
(b) $z = 2e^{i\frac{\pi}{4} + \frac{\pi}{2}k}$, $k = 0, 1, 2, 3$
7. $z = e^{i\frac{\pi}{4}} - 1$ and $z = e^{i\frac{5\pi}{4}} - 1$

Chapter 2

2. (a) 0
(b) $1 + i$
10. (a) differentiable and analytic in $\mathbb{C}$ with derivative $-e^{-x}e^{-iy}$
(b) nowhere differentiable or analytic
(c) differentiable on $\{x + iy \in \mathbb{C} : \ x = y\}$ with derivative $2x$, nowhere analytic
(d) nowhere differentiable or analytic
(e) differentiable and analytic in $\mathbb{C}$ with derivative $-\sin x \cosh y - i \cos x \sinh y$
(f) differentiable at 0 with derivative 0, nowhere analytic
(g) differentiable at 0 with derivative 0, nowhere analytic
(h) differentiable only at i with derivative i, nowhere analytic
(i) differentiable and analytic in $\mathbb{C}$ with derivative $-2iz$

Chapter 3

26. (a) differentiable at 0, nowhere analytic

(b) differentiable and analytic on $\mathbb{C} \setminus \left\{ -1, e^{i\frac{\pi}{3}}, e^{-i\frac{\pi}{3}} \right\}$

(c) differentiable and analytic on $\mathbb{C} \setminus \{ x + iy \in \mathbb{C} : \ x \geq -1, \ y = 2 \}$

(d) nowhere differentiable or analytic

(e) differentiable and analytic on $\mathbb{C} \setminus \{ x + iy \in \mathbb{C} : \ x \leq 3, \ y = 0 \}$

(f) differentiable and analytic in $\mathbb{C}$ (i.e. entire)

27. (a) $z = i$

(b) There is no solution.

(c) $z = \ln \pi + i \left(\frac{\pi}{2} + 2\pi k \right), \ k \in \mathbb{Z}$

(d) $z = \frac{\pi}{2} + 2\pi k \pm 4i, \ k \in \mathbb{Z}$

(e) $z = \frac{\pi}{2} + \pi k, \ k \in \mathbb{Z}$

(f) $z = \pi k, \ k \in \mathbb{Z}$

(g) $z = 2i$

30. $f'(z) = c\, z^{c-1}$

Chapter 4

2. $-2\pi i$

3. (a) $8\pi i$

(b) 0

(c) 0

(d) 0

14. 0

16. $\frac{2\pi}{\sqrt{3}}$

23 0 for $r < |a|$; $2\pi i$ for $r > |a|$

24 0 for $r = 1$; $-\frac{\pi i}{3}$ for $r = 3$; 0 for $r = 5$

Chapter 5

2. (a) 0

(b) $2\pi i$

(c) 0

(d) πi

(e) 0

(f) 0

5. Any simply connected set which does not contain the origin, for example, $\mathbb{C} \setminus (-\infty, 0]$.

Chapter 7

1. (a) divergent

(b) convergent (limit 0)

(c) divergent

(d) convergent (limit $2 - \frac{i}{2}$)

(e) convergent (limit 0)

16. (a) pointwise convergent for $|z| < 1$, uniform for $|z| \leq R$ for any fixed $R < 1$

(b) pointwise and uniformly convergent for $|z| \leq 1$

(c) pointwise convergent for all $z \in H := \{ \mathrm{Re}\, z \geq 0 \}$, uniform on $H \cap \{ |z| \geq R \}$ for any fixed $R > 0$.

18. (a) $\sum_{k\geq 0}(-4)^k z^k$

(b) $\sum_{k\geq 0}\frac{1}{3\cdot 6^k}z^k$

20. (a) $\sum_{k\geq 0}(-1)^k(z-1)^k$

(b) $\sum_{k\geq 1}\frac{(-1)^{k-1}}{k}(z-1)^k$

23. (a) ∞ if $|a| < 1$, 1 if $|a| = 1$, and 0 if $|a| > 1$.

(b) 1

(c) 1 (careful reasoning!)

(d) 1 (careful reasoning!)

Chapter 8

1. (a) $\{z \in \mathbb{C} : \ |z| < 1\}$, $\{z \in \mathbb{C} : \ |z| \leq r\}$ for any $r < 1$

(b) $\mathbb{C}$, $\{z \in \mathbb{C} : \ |z| \leq r\}$ for any r

(c) $\{z \in \mathbb{C} : \ |z - 3| > 1\}$, $\{z \in \mathbb{C} : \ r \leq |z - 3| \leq R\}$ for any $1 < r \leq R$

3. $\sum_{k\geq 0}\frac{e}{k!}(z-1)^k$

10. The maximum is 3 (attained at $z = \pm i$), and the minimum is 1 (attained at $z = \pm 1$).

12. One Laurent series is $\sum_{k\geq 0}(-2)^k(z-1)^{-k-2}$, converging for $|z - 1| > 2$.

13. One Laurent series is $\sum_{k\geq 0}(-2)^k(z-2)^{-k-3}$, converging for $|z - 2| > 2$.

14. One Laurent series is $-3(z+1)^{-1} + 1$, converging for $z \neq -1$.

15. $\frac{1}{\sin z} = z^{-1} + \frac{1}{6}z + \frac{7}{360}z^3 + \dots$

20. (a) $\sum_{k\geq 0}\frac{(-1)^k}{(2k)!}z^{2k-2}$

Chapter 9

5. (a) 0

(b) 1

(c) 4

7. (a) One Laurent series is $\sum_{k\geq -2}\frac{(-1)^k}{4^{k+3}}(z-2)^k$, converging for $0 < |z - 2| < 4$.

(b) $-\frac{\pi i}{8}$

8. (a) $2\pi i$

(b) $\frac{27\pi i}{4}$

(c) $-\frac{2\pi i}{17}$

(d) $\frac{\pi i}{3}$

(e) $2\pi i$

(f) 0

9. (a) $\sum_{k\geq 0}\frac{1}{e\,k!}(z+1)^k$

(b) $\frac{2\pi i}{e\,33!}$

14. (c) $\frac{\pi}{2}$

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